Lecture XXI: Abstract tropical varieties II (& linear spaces)

Last time: Modelled on the Structure Thm, we defined

Def: An abstract tropical variety Σ in $\mathbb{R}^n$ over $\Gamma_{val} = val(K^\times)$ is a pair (Σ, wt)

st
- Σ is a pure Γ_{val}-rat'l polyhedral complex, connected ($d = \dim \Sigma :=$ dim of any mxl cell)
- $wt : \Sigma^{(d)} \longrightarrow \mathbb{Z}_{\geq 0}$ weights (on maximal cells) that make Σ balanced in codim-1.
  $\sigma \longmapsto m_{(\sigma)}$
- We identify (Σ, wt) with any refinement (Σ', wt') where if $\underset{\text{mxl in } \Sigma'}{\sigma'} \subseteq \underset{\text{mxl in } \Sigma}{\sigma} \rightsquigarrow wt'(\sigma') = wt(\sigma)$

Note: Not every abstract trop. variety is realizable, meaning $\Sigma = \mathrm{Trop}(X)$ for some $X \subseteq (K^\times)^n$ variety of $\dim = \dim \Sigma$.

Eg of genus 1 curve of degree 3 in $\mathbb{R}^3$ (last time)

TODAY: Another example (defined in Lectures XVI - XVII) = trop linear spaces.

§1 Tropical Linear spaces ARE abstract trop varieties:

Recall: Trop linear spaces $\longleftrightarrow$ valuated matroids (M, ω).

Eg (Realizable one) [see Lecture XVII]

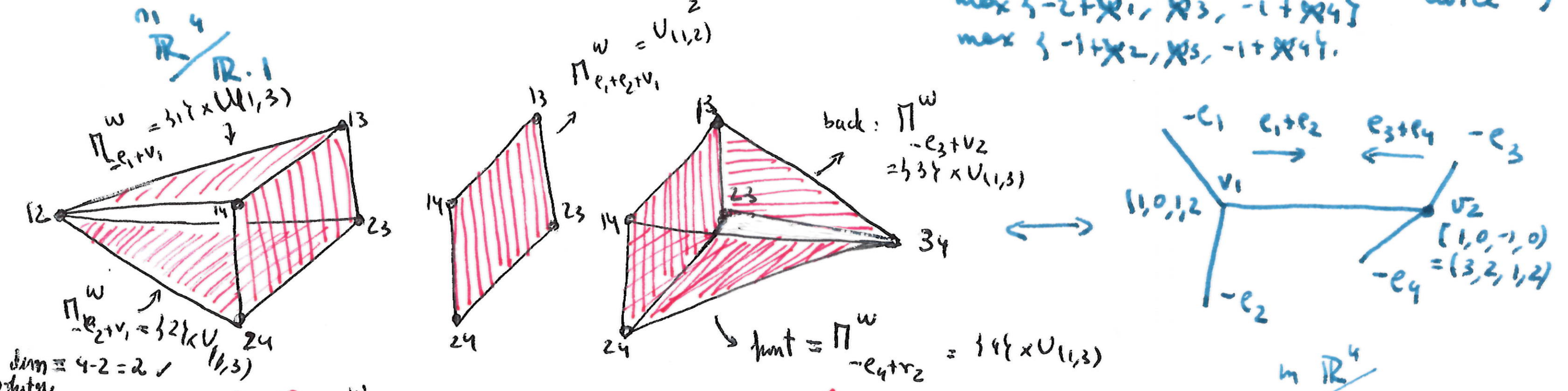
$$V = \text{row space} \left| \begin{array}{cccc} 1 & t & t^2 & t^3 \\ t^3 & t^2 & t & 1 \end{array} \right| \rightsquigarrow \text{matroid } M = U_{(2,4)}$$
$\phantom{V = \text{row space}}$ $x_1 \; x_2 \; x_3 \; x_4$

circuits = 123, 124, 134, 234. $\hookrightarrow$ 4 eqns: $\{p_{ij} X_k - p_{ik} X_j + p_{jk} X_i\}$, (ijk) circuit

$$\underline{P} = \text{Plücker vector} = \underset{12}{(t^2 - t^4}, \underset{13}{t - t^5}, \underset{14}{1 - t^6}, \underset{23}{t - t^5}, \underset{24}{t - t^5}, \underset{34}{t^2 - t^4)}$$

$$\rightsquigarrow \text{tropical Plücker vector} = \omega = \underset{12}{(-2} \;\; \underset{13}{-1} \;\; \underset{14}{0} \;\; \underset{23}{-2} \;\; \underset{24}{-1} \;\; \underset{34}{-2)} \in \mathbb{R}^{\mathcal{B}(M)} / \mathbb{R}\cdot\underline{1}.$$
$$= -val(P)$$

$$\underset{\mathbb{R}^4/\mathbb{R}\cdot 1}{\mathrm{Trop}(V)} = V(\text{tropicalization of circuits}) = \left\{ X : \begin{array}{l} \max\{-1 + X_1, X_2, -1 + X_3\}, \\ \max\{-1 + X_1, X_2, -2 + X_4\} \\ \max\{-2 + X_1, X_3, -1 + X_4\} \\ \max\{-1 + X_2, X_3, -1 + X_4\}. \end{array} \text{ all attained twice} \right\}$$



& filled $= \Pi^{w}_{v_1} = \{12, 13, 14, 23, 24\}$ (connected) $\rightsquigarrow$ dim polytope $= 4-1 = 3$ ✓ & filled $= \Pi^{w}_{v_2} = \{13, 14, 23, 24, 34\}$ (connected) □

• M loopless rank r matroid on $[n]=\{1,\dots,n\}$

Break M into <u>connected components</u>:

$$M = M_1 \oplus \cdots\cdots \oplus M_{s(M)}$$

M_i: connected loopless matroid of E_i of rank r_i
↳ [no coloops]
· not sum of matroids

where $\bigsqcup_{i=1}^{s(M)} E_i = [n]$, $\sum_{i=1}^{s(M)} r_i = r$.

⇝ matroid polytope P_M of dim $= n - s(M)$ in $\mathbb{R}^n$ ($= r-1$ if M is connected)

$$P_M = \text{convex hull } \{\sum_{b\in B} e_b : B \in \mathcal{B}(M)\} = P_{M_1} \times \cdots \times P_{M_{s(M)}} \text{ in } \mathbb{R}^n.$$

• w = tropical Plücker vector in Dr_M, ie a pt in $\mathbb{R}^{\mathcal{B}(M)}/\mathbb{R}\cdot\mathbb{1}$ satisfying <u>ALL</u> the tropical Plücker relns: $\operatorname{tr} \mathcal{P}l_{\sigma,\tau} = \max_j \{w_{\sigma\cup j} + w_{\tau\smallsetminus j} : \sigma\cup j, \tau\smallsetminus j \in \mathcal{B}(M)\}$
$|\sigma| = r-1$
$|\tau| = r+1$
$|\tau\smallsetminus\sigma| \geq 3$

$\boxed{\text{Why? } w \text{ determined a reg subdivision of } P_M \text{ into matroid polytopes}}$

[Lecture XVII]: $L_w = \bigcap_{\substack{|\tau|=r+1 \\ \operatorname{rk}\tau = r}} L_\tau(w)$ where $L_\tau(w) = V\left(\bigoplus_{j\in\tau} w_{\tau\smallsetminus j} \odot \underline{x_j}\right) \subseteq \mathbb{R}^n/\mathbb{R}\cdot\mathbb{1}$

↳ (so a circuit!)

Prop [Speyer] $L_w = \{u \in \mathbb{R}^n/\mathbb{R}\cdot\mathbb{1} : \text{the matroid } M_u^w \text{ is loopless}\}$

where $M_u^w = \{\sigma \in \mathcal{B}(M) : w_\sigma - \sum_{j\in\sigma} u_j \text{ is maximal}\}$

[for $w \equiv 0$ we get matroid M_u^0 of loopless faces dual to $-u$. ⇝ same def as $\widetilde{\mathcal{B}}(M)$]

These are the bases indexing vertices in the subdivision of P_M induced by w in the face dual to $-u$. (i.e. dual to $(-u,1)$ a face of $UF((P_M)_w)$ in the notation of Lecture VI)

Example (cont.) $u = v_1 = [1,0,1,2]$

	w_σ	$-\sum_{j\in\sigma} u_j$
12	-2	-1
13	-1	-2
14	0	-3
23	-2	-1
24	-1	-2
34	-2	-2

$\left.\right\}$ $= -3$ MAX. (rows 12–24)

$= -4$ (row 34)

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Claim: $(M^w_u)_v$ = loopless matroid dual to v in $P_{M^w_u}$ = bases in M^w_u with maximal $(-v)$ weight

$\quad\|$ for $0<\varepsilon<<1$

$M^w)_{u+\varepsilon v}$ = ——————— $u+\varepsilon v$ in P_M.

[Analog result for iterative initial forms]

Corollary: Given a cell σ in L_w & $u \in \operatorname{rel\,int}(\sigma)$:

$$\operatorname{Star}_{L_w}(\sigma) = \underbrace{\operatorname{Trop}(M^w_u)}_{\text{[trivial vals]}} = \widetilde{\mathcal{B}}(M^w_u) = \{v \in \mathbb{R}^n/\mathbb{R}\cdot\mathbb{1} : (M^w_u)_v \text{ loopless}\}$$

$\Rightarrow$ Using this we can show that L_w satisfies the defining properties of an abstract trop. variety

Theorem Fix a loopless rank r matroid M on $[n]$ & $w \in Dr_M$. The tropical linear space L_w is a pure $(r-1)$-dim'l rational polyhedral complex in $\mathbb{R}^n/\mathbb{R}\cdot 1$ (Γ_{val}-rat'l if $w \in \Gamma_{val}^n/\mathbb{R}\cdot 1$), balanced in codim-1 with constant multipl. $=1$.

Furthermore, it is contractible & its recession fan $= \operatorname{Trop}(M) =$ Bergman fan of M.

In addition, it has degree $=1$, meaning $L_w \cap (p + \operatorname{Trop}(U_{(n-r+1,n)}))$ is transverse & gives one pt.

$\hookrightarrow$ generic pt $\qquad\hookrightarrow$ generic trop $(n-r)$-dim'l plane in $\mathbb{R}^n/\mathbb{R}\cdot 1$

- Contractible = uses tropical convexity; For deg $=1$ see [Fink].

Proof We prove the pureness, balancing & recession fan properties for a connected matroid M (If $M = M_1 \oplus \cdots \oplus M_s$, set $Dr_M = Dr_{M_1} \times \cdots \times Dr_{M_s}$ & $L_w = L_{w_1} \times \cdots \times L_{w_s}$ "$w_i := w|_{M_i}$" so the result for M follows from the M_i's)

(1) Recession fan [similar to Lecture XX]

- $\operatorname{rec}(L_\mathcal{C}(w)) \underset{\text{def.}}{=} L_\mathcal{C}(0) \rightarrow$ defined by the circuit $\mathcal{C}$ for $w \neq 0$ & $w=0$.

$$\Rightarrow \operatorname{rec}(L_w) \underset{\substack{\uparrow \\ \text{nice } \cap}}{=} \bigcap_{\substack{|\mathcal{C}|=r+1 \\ \operatorname{rk}\mathcal{C}=r}} L_\mathcal{C}(0) = \widetilde{\mathcal{B}}(M) \quad \text{(the circuits of } M \text{ cut out } \widetilde{\mathcal{B}}(M))$$

(2) Pure of dim $= r-1$ & rat'l polyhedral complex

- $\dim P_M = n-1$ & all mx'l pieces in the subdivision of P_M induced by w have dim $= n-1$, so they are matroid polytopes of connected ~~rank~~ $=r$ matroid on $[n]$ with no loops. The matroid is $M^w_{\underline{u}}$ for some $u_{\text{Exstix}=\sigma}$ & $\operatorname{Star}_{L_w}(\sigma) = \widetilde{\mathcal{B}}(M^w_u)$.

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- M^w_u (conn) rank r on $[n]$ $\Rightarrow$ $\tilde{\mathcal{B}}(M^w_u)$ is pure of dim $= r-1$. & rat'l polyhedral complex.

Why? cells μ in Bergman fan $\longleftrightarrow$ flags of flats $\mathcal{F}_\mu = \{\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_s \subsetneq F_{s+1} = [n]\}$
$\dim \mu = s$ $\qquad$ rk 0, rk $= i_1$, rk $= i_s$, rk $= r$.

via $\mu = (\mathbb{R}_{\geq 0}\langle -e_{F_1}, -e_{F_2}, \ldots, -e_{F_s}\rangle + \mathbb{R}\cdot\underline{1}) \,/\, \mathbb{R}\cdot\underline{1}$

where $e_F := \sum_{i\in F} e_i \in \mathbb{R}^n$ (indicator vector of the flat)

Mxl cells $\longleftrightarrow$ mxl flats. (at each step the rank jump exactly by 1) So $\underline{\dim = r-1}$.

(3) Balancing:

Using corollary, it suffices to show that Bergman fans of loopless connected matroids are balanced.

We prove it for $\tilde{\mathcal{B}}(M)$.

- For this, we need the following "covering property".

Prop: If F is a flat of M, then the flats of M that cover F partition the elements of $[n]\setminus F$.
$\hookrightarrow \begin{cases} G \supset F \text{ flat} \\ \operatorname{rk} G = \operatorname{rk} F + 1 \end{cases}$

- Codim-1 cone τ in $\tilde{\mathcal{B}}(M) \longleftrightarrow$ flag $\mathcal{F} = \{\underbrace{\emptyset = F_0}_{\operatorname{rk}=0} \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{r-2} \subsetneq \underbrace{F_{r-1}}_{\operatorname{rk}=r} = [n]\}$

Note: . All jump in rank ≥ 1
$\quad$. Total rank r & $(r-1)$ steps $\Big\}$ $\Rightarrow$ Exactly $\underline{\text{ONE}}$ rank 2 jump, say at $\underbrace{F_\ell \subsetneq F_{\ell+1}}_{\text{rank } 2}$ for some ℓ.

Mxl cells $\sigma \supsetneq \tau$ $\longleftrightarrow$ completions of $\mathcal{F}$ to a full flag $\longleftrightarrow$ $F_\ell \overset{\text{covering}}{\subsetneq} \boxed{G_\sigma} \subsetneq F_{\ell+1}$

- We claim: $\sum_{\substack{\sigma \supsetneq \tau \\ \sigma = \mathcal{F}\cup\{G_\sigma\}}} e_{G_\sigma} = \underbrace{e_{F_{\ell+1}} + (s-1)e_{F_\ell}}_{\text{its negative in } \tau.}$ for $s = \#\{\text{mxl cells } \sigma \supsetneq \tau\}$

$\therefore$ $\tilde{\mathcal{B}}(M)$ balanced at τ: $\left(\dfrac{\sigma + \mathbb{R}\langle\tau\rangle}{\mathbb{R}\langle\tau\rangle} \simeq \mathbb{R}_{\geq 0}\langle e_{G_\sigma}\rangle. \right)$

Proof (Claim): Construct a matroid M_ℓ by deleting all elements $\underline{\text{NOT}}$ in $F_{\ell+1}$ from bases of M.

Flats of M_ℓ $\longleftrightarrow$ flats of M contained in $F_{\ell+1}$.

By the "covering property" for M_ℓ: every element of $F_{\ell+1}\setminus F_\ell$ is contained in EXACTLY one $G_\sigma \setminus F_\ell$, so $\bigsqcup_{\sigma \supsetneq \tau} (G_\sigma \setminus F_\ell) = F_{\ell+1}\setminus F_\ell$ & claim follows $\square$.