Lecture XXII: Chow polytopes

TODAY: Discuss Chow varieties of subvarieties X of $\mathbb{P}^{n-1}$ & relate them to $\mathrm{Trop}(X \cap (K^*)^n / K^*)$

- Hypersurfaces in certain Grassmannians $\rightsquigarrow$ Newton polytope of Chow variety $=:$ Chow polytope
- The Chow polytope generalizes: (1) Newton polytope of hypersurfaces.
  (2) Matroid polytopes of linear spaces.
- Upshot: • hard to compute algebraically ($\sim$ resultants, discriminants)
  - Chow polytopes are easier & we can use tropical geometry (combinatorial!)
  - Chow polytopes record toric degenerations of X.

§1 Chow varieties [Chow-van der Waerden (1937)]

Def: Fix $X \subset \mathbb{P}^{n-1}$ of dimension $k-1$ & degree d (irreducible or not). We identify X with a pt in the Chow variety $\mathcal{G}(k,d,n) = \{$all effective algebraic cycles in $\mathbb{P}^{n-1}$ of dimension $k-1$ & degree $d\}$

Here: a cycle is a formal sum $X = \sum_i m_i X_i$: • $X_i \subseteq \mathbb{P}^{n-1}$ irred, dim $= k-1$,
• $m_i \in \mathbb{Z}_{\geq 0}$ (effective)

We define $\deg(X) = \sum_i m_i \deg(X_i) = d$.

Example: (1) $\mathcal{G}(k,1,n) =$ degree 1 cycles $= Gr(k,n)$

(2) $\mathcal{G}(n-1,d,n) =$ degree d hypersurfaces in $\mathbb{P}^{n-1} = \mathbb{P}(K^{\binom{n-1+d}{d}})$

$=$ degree d polynomials in n variables $/ K^*$.

- $\mathcal{G}(k,d,n)$ is a projective variety, so we can embed it in $\mathbb{P}^N$ for some N.

For this, we use the Grassmannian $Gr(n-k,n)$ & its coordinate ring

$$K[Gr(n-k,n)] = \frac{K[p_I : I \subset \binom{[n]}{n-k}]}{I_{n-k,n}} = \bigoplus_d B_d$$

$\hookrightarrow$ homog degree d pieces $\simeq \mathbb{A}^{N_d+1}$

dim $= (n-k)k$

- We define an embedding $\varphi : \mathcal{G}(k,d,n) \longrightarrow \mathbb{P}(B_d)$ as follows:

- On irreducible cycles X:

$$Z(X) = \{L \in Gr(n-k,n) : L \cap X \neq \emptyset\}$$

(proj dim $L = n-k-1$)

For generic L of dim $n-k$, $L \cap X$ is a collection of d ($= \deg X$) pts (counted with mult.)
so a generic L of dim $n-k-1$ doesn't meet X! So $Z(X)$ contains special elements

Theorem: $Z(X) \subseteq Gr(n-k,n)$ is an irreducible hypersurface & degree $= d$

FACT: $Gr(n-k,n)$ locally $\simeq \mathbb{A}^{k(n-k)}$ so $Z(X)$ is cut out by one equation in $\{p_I\}$

We call the polynomial defining $\mathcal{Z}(X)$ in $\mathcal{B}_d$ the <u>Chow form of X</u> & denote it by R_X.

<u>Proof</u> (Theorem): Use classical trick = incidence varieties

$$B(X) = \{(x,L) : x \in X,\ L \in Gr(n-k,n),\ x \in L\}$$

$$\begin{tikzcd} & B(X) \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{Z}(X) & & X \end{tikzcd} \qquad (x,L) \mapsto L, \qquad (x,L) \mapsto x$$

- <u>Know</u>: generic L in $Gr(n-k+1,n)$ meets X at d pts $\Rightarrow$ an $L' \in \mathcal{Z}(X)$ meets X at exactly 1 pt (Take L & cut out one of the d pts by a single generic eqn.)

  <u>Conclusion</u>: q is a birational isomorphism.

- p is a Grassmannian fibration: $p^{-1}(x) \simeq Gr(n-k-1, n-1)$ irred!
- Since X is irreducible, so is $B(X)$ (Fibers & base are irred) $\Rightarrow$ so is $\mathcal{Z}(X)$ (birat iso)
- Dimension count:

$$\dim \mathcal{Z}(X) = \dim B(X) = \underbrace{(n-k-1)k}_{\text{dim fibers}} + \underbrace{(k-1)}_{\text{dim base}} = k(n-k)-1 = \dim Gr(n-k,n) - 1.$$

- Degree computation:
  - Choose a generic flag $N \subset M \subset \mathbb{P}^{n-1}$ of projective subspaces, $\dim N = n-k-2$, $\dim M = n-k$.
  - Count: $\#\{(n-k-1)\text{-dim'l subspace } L \in \mathcal{Z}(X) : N \subset L \subset M\} = \deg(\mathcal{Z}(X))$ $(*)$
  - $\deg X = d \rightsquigarrow M \cap X = d$ pts, called $x_1, \ldots, x_d$.

    $\Rightarrow L_i = \mathbb{P}\langle \text{span}(N, x_i) \rangle$ are the only Ls we can find in $(*)$ $\Rightarrow$ count gives d such Ls. $\square$

<u>Remark</u>: The map $X \longmapsto \mathcal{Z}(X)$ is injective ($p \in \mathbb{P}^{n-1}$ lies in X if and only if ($p \in L$ & L of dim $n-k-1 \Rightarrow L \in \mathcal{Z}(X)$))

- If X is not irreducible, write $X = \sum_i m_i X_i \in G(k,d,n)$, we define

$$R_X = \prod_i \underset{\substack{\cap \\ \mathcal{B}_{d_i}}}{R_{X_i}^{m_i}} \in \mathcal{B}(d) \qquad [d = \sum_i m_i d_i]$$

<u>Example</u> (0) $X = \{pt\}$ $(k=1)$ $G(n-k,n) = (\mathbb{P}^{n-1})^\vee$ & $\mathcal{Z}(X)$ = hyperplane dual to p.

(1) $X \subset \mathbb{P}^3$ curve, $\mathcal{Z}(X) = \{\text{lines in } \mathbb{P}^3 \text{ meeting } X\}$

3c Chow Polytopes:

- $GL_n(K) \circlearrowright G(k,d,n)$ by the action of $GL_n(K)$ on $\mathbb{P}^{n-1}$!

$$\begin{array}{c} GL_n(K) \\ \cup \\ (K^*)^n = \text{max'l torus} \end{array} \text{ (diagonal matrices)} \rightsquigarrow \text{character lattice} \simeq \underset{M = \mathrm{Hom}((K^*)^n, K^*)}{\mathbb{Z}^n}$$

Def: The <u>Chow polytope</u> $Ch(X)$ of $X \in G(k,d,n)$ is the convex hull of the exponents of R_X (= Newton polytope of R_X), viewed in $\mathbb{R}^n$ (exponents of R_X viewed in $\mathbb{Z}^n$ via <u>T-weights</u>)

Q: How to construct $Ch(X)$ in $\mathbb{R}^n$?

(1) X irreducible

Write R_X as a homogeneous degree d polynomial in <u>dual Plücker coordinates</u> $[\sigma] = [i_1 \cdots i_k]$ of $Gr(n-k,n)$ (Here $\sigma = \{i_1 < \cdots < i_k\} \subset [n]$) $Gr(n-k,n)^\vee = Gr(k,n)$

(E.g. $X \in Gr(k,n)$, then $R_X = \sum\limits_{i_1 \cdots i_k} \xi_{i_1 \cdots i_k} [i_1 \cdots i_k]$)

$\{\xi_{i_1 \cdots i_k}\}$ Pl coords

- Each monomial $\prod\limits_\sigma [\sigma]^{m_\sigma}$ in R_X has <u>T-weight</u> $\sum\limits_\sigma m_\sigma \sum\limits_{i \in \sigma} e_i \in \mathbb{Z}^n$

<u>Conclusion</u>: $Ch(X) \subseteq d\,\Delta(k,n)$ $\quad [\Delta(k,n) = \{\underline{x} \in \mathbb{R}^n : 0 \leq x_i \leq 1, \ \sum x_i = k\}$

(2) $X = \sum m_i X_i \rightsquigarrow Ch(X) = \sum \underbrace{m_i\, Ch(X_i)}_{\text{dilation}}$ (Minkowski sum)

<u>Example 0</u>: $X \subseteq \mathbb{P}^n$ hypersurface $V(F=0) \Rightarrow R_X = F([2 \cdots n], -[13\cdots n], \ldots, (-1)^n [12 \cdots n-1])$;

<u>Example 1</u>: $X \in Gr(2,4)$ = line through $(a_1:a_2:a_3:a_4)$ & $(b_1:b_2:b_3:b_4)$ in $\mathbb{P}^3$

$\rightsquigarrow$ Write Plücker coords as $([12]:[13]:[14]:[23]:[24]:[34])$ for $L \in Gr(2,4)$

$X \cap L \neq \emptyset \iff R_X = (a_1b_2 - a_2b_1)[12] + (a_1b_3 - a_3b_1)[13] + \cdots + (a_3b_4 - a_4b_3)[34]$

Plücker coords of X

<u>Example 2</u> [Twisted cubic curve] $X = \{(s^3 : s^2t : st^2 : t^3) \in \mathbb{P}^3 : (s:t) \in \mathbb{P}^1\}$

$X = \{(x_1 : \cdots : x_4) \in \mathbb{P}^3 : x_1x_3 - x_2^2 = x_1x_4 - x_2x_3 = x_2x_4 - x_3^2 = 0\} \rightsquigarrow n = 4, \ d = 3, \ k = 2$

$$(*) \quad R_X = \det \begin{vmatrix} u_1 & u_2 & u_3 & u_4 & 0 & 0 \\ & u_1 & u_2 & u_3 & u_4 & 0 \\ 0 & 0 & u_1 & u_2 & u_3 & u_4 \\ v_1 & \cdots & & v_4 & \cdots & 0 \\ 0 & 0 & v_1 & \cdots & & v_4 \end{vmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} 4-1=3 \\ \left.\vphantom{\begin{matrix} a \\ a \end{matrix}}\right\} 4-1=3 \end{array}$$

Write $L \in Gr(2,4)$ as $\ker \begin{pmatrix} u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \end{pmatrix}$

Then $X \cap L \neq \emptyset \iff \exists (s:t) \in \mathbb{P}^1$
$$u_1 s^3 + u_2 s^2t + u_3 st^2 + u_4 t^3 = 0$$
$$v_1 s^3 + v_2 s^2t + v_3 st^2 + v_4 t^3 = 0$$

R_X = Sylvester resultant in $[ij] = u_iv_j - u_jv_i = \det \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}$ (*)

$$= -[14]^3 - [14]^2[23] + 2[13][14][24] - [12][24]^2 - [13]^2[34] + \underbrace{[12][14][34] + [12][23][34]}_{\text{Plücker coords of } L} = \det \begin{pmatrix} [12] & [13] & [14] \\ [13] & [14]+[23] & [24] \\ [14] & [24] & [34] \end{pmatrix}$$