Lecture XXIII: Chow polytopes & Tropicalization

Recall: The Chow variety $G(k,d,n) = \{$ all effective algebraic cycles in $\mathbb{P}^{n-1}$ of dim $k-1$ & degree d $\}$

Formally $X = \sum_{i=1}^{N} m_i X_i$: $X_i \subseteq \mathbb{P}^{n-1}$ irred, dim $=k-1$; $\sum_{i=1}^{N} m_i \deg(X_i) = d$. $m_i \in \mathbb{Z}_{>0}$ $\forall i$

$\rightsquigarrow$ If X irred, we define $Z(X) = \{L \in Gr(n-k,n) : L \cap X \neq \emptyset\}$ & showed it's a hypersurface of $Gr(n-k,n)$, irreducible of degree $= d$, call $R_X \subseteq K[p_I]_{I \in \binom{[n]}{k}}$ the defining equation (Chow form), in fact $R_X \in \mathbb{P}(B_d)$ where $K[Gr(n-k,n)] = \bigoplus_{d\geq 0} B_d$

- For general $X = \sum m_i X_i$, we set $R_X = \prod_{i=1}^{N} R_{X_i}^{m_i}$ $R_{X_i} \in \mathbb{P}(B_{d_i})$, $\sum m_i d_i = d$. [$\deg p_I = 1$ $\forall I$.]

Q Behavior under field extensions? $L|K$.

. If X irred over K but factors over L. $\Rightarrow R_X \subseteq K[Gr(n-k,n)]$ irreducible but $R_X = \prod_{i=1}^{N} R_{X_i}^{m_i}$ & $R_{X_i} \subseteq L[Gr(n-k,n)]$.

Example: $K = \mathbb{Q}$, $L = \mathbb{Q}(\omega)$ ω primitive 3^{rd} root of unity.

$X = X_{\mathbb{Q}} = V(x+y+z,\ y^2 - xz$ but $X_L = \{(1:\omega:\omega^2), (1:\omega^2:\omega)\}$

irred scheme over $\mathbb{P}^2_{\mathbb{Q}}$ 2 components / $\mathbb{P}^2_L$. $(m_i = 1)$

$$R_X = [0]^2 - [0][1] + [1]^2 - [0][2] - [1][2] + [2]^2$$
$$= ([0] + \omega[1] + \omega^2[2])([0] + \omega^2[1] + \omega[2])$$

§1 Linear transformations & Toric degenerations

- Natural action: $GL(n,K) \circlearrowright K^n$ & $\mathbb{P}^{n-1}$, so given $X \subseteq \mathbb{P}^{n-1}$ & $g \in GL(n,K)$, we have $g(X) \subseteq \mathbb{P}^{n-1}$ $g(X) \simeq X$

Likewise $GL(n,K) \circlearrowright \mathbb{P}^{\binom{n}{k}-1} \simeq (\Lambda^{n-k}(K^n))$ $\rightsquigarrow$ acts on $Gr(k,n)$

Furthermore $GL(n,K) \circlearrowright \mathbb{P}(B_d)$ $\forall d \in \mathbb{Z}_{\geq 0}$. $\rightsquigarrow$ can act on R_X (Chow form)

Prop: For $X \subseteq \mathbb{P}^{n-1}$, $g \in GL(n,K)$ we have $R_{g(X)} = g * R_X$

Pf/ $\{L : gX \cap L \neq \emptyset\} = \{L : X \cap g^{-1}L \neq \emptyset\} = g \circ \{L' : X \cap L' \neq \emptyset\}$ □.

- Torus action $T \simeq (K^*)^n$ invertible diag matrices in $GL(n,K)$. ; char. lattice $M \simeq \mathbb{Z}^n$

$\rightsquigarrow$ $\overline{TX} \subset G(k,d,n) \subset \mathbb{P}(B_d)$ is a projective Toric variety with torus T & face poset of $Ch(X) =$ Poset of T-orbits of $\overline{TX}$ (inclusion of orbit closures)

[Kapranov-Sturmfels-Zelevinsky]

Thm: Fan of $\overline{TX}$ = normal fan of $Ch(X)$ in $\mathbb{R}^n$

Here, $Ch(X)$ is the Chow polytope of X. Each monomial $\prod_{\sigma} [\sigma]^{m_\sigma}$ in R_X has T-weight $\sum_{\sigma} m_\sigma \sum_{i\in\sigma} e_i$

Plücker coord in $Gr(k,n)$ $\sigma = \{i_1, \ldots, i_k\}$

- One parameter subgroups $:\longleftrightarrow \mathrm{Hom}(k^*, (k^*)^n) = N$

Given $w = (w_1, \dots, w_n) \in \mathbb{Z}^n$ (N) & $t \in k^*$ we define $\lambda_w : k^* \longrightarrow (k^*)^n$, $t \longmapsto \mathrm{diag}(t^{w_1}, \dots, t^{w_n})$

Remark $\lambda_w(t) \circ R_X = R_{\lambda_w(t)\circ X}$ is obtained by rescaling each bracket $[i_1 \cdots i_k] \mapsto t^{w_{i_1}+\dots+w_{i_k}}[i_1 \cdots i_k]$

- From the ideal perspective $I(\lambda_w(t)\circ X) = \langle f(x_1 t^{-w_1}, \dots, x_n t^{-w_n}) : f \in I(X) \rangle$

Def A cycle Y is a toric degeneration of X if $Y = \lim\limits_{t\to 0} \underbrace{\lambda_w(t)\, X}_{\text{in } X_{k[t^{\pm}]}}$ for some $w \in M$ ↳ resembles w initial computations for trivial vals!

The defining ideal of Y is the w-initial ideal of $I(X)$

Remark: On Chow forms, the toric degeneration is given as the leading terms of $R_{\lambda_w(t)X}$ with respect to the weights: $wt[\sigma] = -w_\sigma$ $\quad (w_\sigma = \sum_{i\in\sigma} w_i \in \mathbb{Z})$.

Example (twisted cubic) $X = \{\underline{x} \in \mathbb{P}^3 : \overset{f_1}{x_1x_3 - x_2^2} = \overset{f_2}{x_1x_4 - x_2x_3} = \overset{f_3}{x_2x_4 - x_3^2} = 0\}$

Take $w = (3,1,0,0) \rightsquigarrow$

$$R_X = \underset{(-9)}{-[14]^3} - \underset{(-7)}{[14]^2[23]} + \underset{(-7)}{2[13][14][24]} - \underset{(-6)}{[12][24]^2} - \underset{(-6)}{[13]^2[34]} + \underset{(-7)}{[12][14][34]} + \underset{(-5)}{\underline{[12][23][34]}}$$

$f_1(\lambda_w(t)\,\underline{x}) = t^{-3}x_1x_3 - t^{-2}x_2^2 = t^{-3}(x_1x_3 - t x_2) \equiv (x_1x_3 - t x_2) \xrightarrow[t\to 0]{} \underset{\mathrm{in}_w(f_1)}{\overset{\shortparallel}{x_1x_3}}$

↳ defined over $K[t^{\pm}]$

similarly $f_2(\lambda_w(t)\,\underline{x}) = t^{-3}x_1x_4 - t^{-1}x_2x_3 = t^{-3}(x_1x_4 - t^2x_2x_3) \equiv (x_1x_4 - t^2x_2x_3) \xrightarrow[t\to 0]{} \underset{\mathrm{in}_w(f_2)}{\overset{\shortparallel}{x_1x_4}}$

$f_3(\lambda_w(t)\,\underline{x}) = t^{-1}x_2x_4 - x_3^2 = t^{-1}(x_2x_4 - t x_3^2) \equiv (x_2x_4 - t x_3^2) \xrightarrow[t\to 0]{} \underset{\mathrm{in}_w(f_3)}{\overset{\shortparallel}{x_1x_4}}$

A Gröbner basis calculation shows $\mathrm{in}_w I = \langle \mathrm{in}_w f_1, \mathrm{in}_w f_2, \mathrm{in}_w f_3 \rangle$
$= \langle x_1x_3, x_1x_4, x_2x_4 \rangle = \langle x_1, x_2\rangle \cap \langle x_1, x_4\rangle \cap \langle x_3, x_4 \rangle$

$\mathrm{in}_{-w_\sigma} R_X = [12][23][34]$ = Chow form of 3 coordinate lines in $\mathbb{P}^3$ [primary decomposition ↗ union of 3 coord lines]

So Y = union of 3 coord lines is the toric degeneration of X wrt $\lambda_w(t) = \mathrm{diag}(t^3, t, 1, 1)$

Here lines are L_{12}, L_{23} & L_{34} (brackets of $\mathrm{in}_{-w_\sigma} R_X$ indicate the defining ideal) □

Remark 2: Extremal toric degenerations of X (T-invariant ones) $\xleftrightarrow{1\text{-to-}1}$ Vertices of $Ch(X)$

Write $L_\sigma = \mathbb{P}\{x_i = 0, i \in \sigma^c\} \subseteq \mathbb{P}^{n-1}$ $\quad \dim L_\sigma = |\sigma| - 1$. $\quad$ ($[\sigma]$ = variables in R_X)

Thm [KSZ] Fix $X \in G(k,d,n)$ irreducible. The vertices of $Ch(X)$ correspond to the toric degenerations of X of the form $\sum_{|\sigma|=k} m_\sigma L_\sigma$ $\quad$ (& vertex $= \sum_\sigma m_\sigma \sum_{i\in\sigma} e_i$)

- Example above $[12]$ gives $L_{12} = \langle x_3, x_4\rangle$, etc so $Y = L_{34} \cup L_{23} \cup L_{12}$.

Prop: If $X \in G(k,d,n)$ is irreducible, $\dim Ch(X) = \underset{n}{\underset{\shortparallel}{\dim T}} - \dim T_X$

where $T_X = \{g \in T : gX = X\}$.

§2 Tropicalization & Chow polytopes:

[Dickenstein-Feichtner-Sturmfels : trivial valn
[Fink : arbitrary valuations]

- Take $X \in G(k,d,n)$ & consider $\Sigma = \mathrm{Trop}\left(X \cap \frac{(K^*)^n}{K^*}\right)$ & $\Sigma' = \mathrm{Trop}(\mathrm{Cone}(X) \cap (K^*)^n)$
- Lecture XX: Used Ray-Shooting algorithm to go from Σ' to the Newton subdivision when $k = n-1$ (trop hypersurfaces are realizable).

- Pick generic point in connected component of $\mathbb{R}^n \smallsetminus \Sigma' = C_v$ for v a vertex in N Subd.(X)
- Get $v = (v_1, \ldots, v_n)$ by computing $(p + \mathbb{R}_{\geq 0}\langle e_j\rangle \cap \Sigma') = \{q_1^{(j)}, \ldots, q_{s(j)}^{(j)}\}$

where $q_i^{(j)} \in$ maxl cell $\sigma_i^{(j)} \subsetneq \Sigma'$

$$v_j = \sum_{i=1}^{s(j)} \mathrm{wt}(q_i^{(j)}) \qquad \mathrm{wt}(q_i^{(j)}) = m(\sigma_i^{(j)}) \left|\det(w_1^{(j)} \cdots w_{n-1}^{(j)}\ -e_j)\right|$$

where $\mathrm{Star}_{\Sigma'}\sigma_i^{(j)} \cap \mathbb{Z}^n = \mathbb{Z}\langle w_1^{(j)}, \ldots, w_{n-1}^{(j)}\rangle$

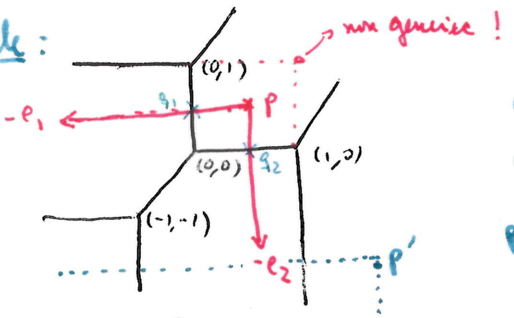
- Edges in the subdivision: (v, v') iff $\overline{C_v} \cap \overline{C_{v'}}$ in codimension 1.
- Weights inducing subdivision of the Newton polytope $\mathcal{P}$
  - Fix a vertex $v_0 \in V(\mathcal{P})$ & set $w_{v_0} = 0$.
  - Any vertex p in Σ' $\longleftrightarrow$ maxl face $\mathcal{F}$ of the subdivision of $\mathcal{P}$. with vertices $\{v_1, \ldots v_s\}$

→ Use linear algebra to get $w_v + \langle v, p\rangle = w_{v'} + \langle v', p\rangle \quad \forall v, v' \in V(\mathcal{F})$ starting from a face containing v_0.

→ . Connectedness (in codim-1) ensure we get all values $(w_v)_{v \in V(\mathcal{P}_{\text{subd}})}$ in this way

. Balancing ensures there are no contradictions & the system is consistent with a unique solution once $w_{v_0} = 0$ is determined

Example:



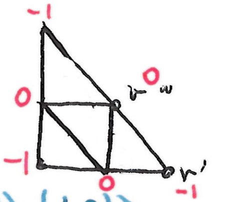
$\mathrm{wt}(q_1) = 1 \det|e_2, -e_1| = 1$

$\mathrm{wt}(q_2) = 1 \det|e_1, -e_2| = 1$

$\Rightarrow v = [1,1]$

p' gives $v' = [2,0]$, etc

We get



Set $\mathrm{wt}(v) = w_{1,1} = 0 \rightsquigarrow w_{1,0} + \underset{1}{\langle(1,0),(1,0)\rangle} = w_{2,0} + \underset{2}{\langle(2,0),(1,0)\rangle} = \underset{\boxed{0}}{w_{1,1}} + \underbrace{\langle(1,1),(1,0)\rangle}_{1}$

$p = (1,0) \in \Sigma'$

so $w_{1,0} = 0$, $w_{2,0} = -1$

Next $p' = (0,0)$ $\quad \underset{0}{w_{1,1}} + \langle(1,1),(0,0)\rangle = \underset{0}{w_{1,0}} + 0 = w_{0,1} + 0 \Rightarrow w_{0,1} = 0$

Next $p'' = (-1,-1)$ $\quad w_{0,0} + 0 = \underset{0}{w_{1,0}} + \underset{-1}{\langle(1,0),(-1,-1)\rangle} = \underset{0}{w_{0,1}} + \underset{-1}{\langle(0,1),(-1,-1)\rangle} \Rightarrow w_{0,0} = -1$

Finally $p''' = (0,1)$ $\quad w_{0,2} + \underset{2}{\langle(0,2),(0,1)\rangle} = \underset{0}{w_{0,1}} + \underset{1}{\langle(0,1),(0,1)\rangle} = \underset{0}{w_{1,1}} + \underset{1}{\langle(1,1),(0,1)\rangle}$

$\Rightarrow w_{0,2} = -1$

Q: What about lower values of k? Replace ray-shooting by Orthant-shooting

Answer: Get a subdivision of $Ch(X)$ induced by $w_i = val(\sigma_i)$ $(i=1,\dots,m)$ where R_X is defined over $k[\sigma_1^{\pm},\dots,\sigma_m^{\pm}] \subseteq K$ subfield. $\sigma_i :=$ coeff of i^{th} monomial [$m=$ total # of monomials]

Thm [Orthant shooting] For $w \in \mathbb{R}^n$ generic vector, a prime ideal $P_\sigma = \langle x_i : i \in \sigma \rangle$ $(|\sigma| = n-k)$ is associated to the initial monomial ideal $in_\omega(I(X)) \subset K[x_1,\dots,x_n]$ if and only if the cone $w + \mathbb{R}_{\geq 0}\{-e_{\sigma_1},\dots,-e_{\sigma_{n-k}}\}$ meets $Trop(Cone X \cap (K^*)^n)$. The number of (finite) intersections $\{p_1 \cdots p_s\}$ counted with multiplicity $\sum_{i=1}^{s} wt(p_i)$, where $wt(p_i) = mult(\sigma_i) \cdot \det\left| v_1 \cdots v_k \ -e_{\sigma_1} \cdots -e_{\sigma_{n-k}} \right|$ if $p_i \in rel\,int(\sigma_i)$, σ_i max'l cell in Σ', $Star_{\Sigma'}\sigma_i \cap \mathbb{Z}^n = \mathbb{Z}\langle v_1,\dots,v_k\rangle$

is the multiplicity of the monomial ideal $in_\omega I(X)$ along P_σ.
(= exponent of $[\sigma]$ in monomial in R_X dual to w)

Example on next slide.

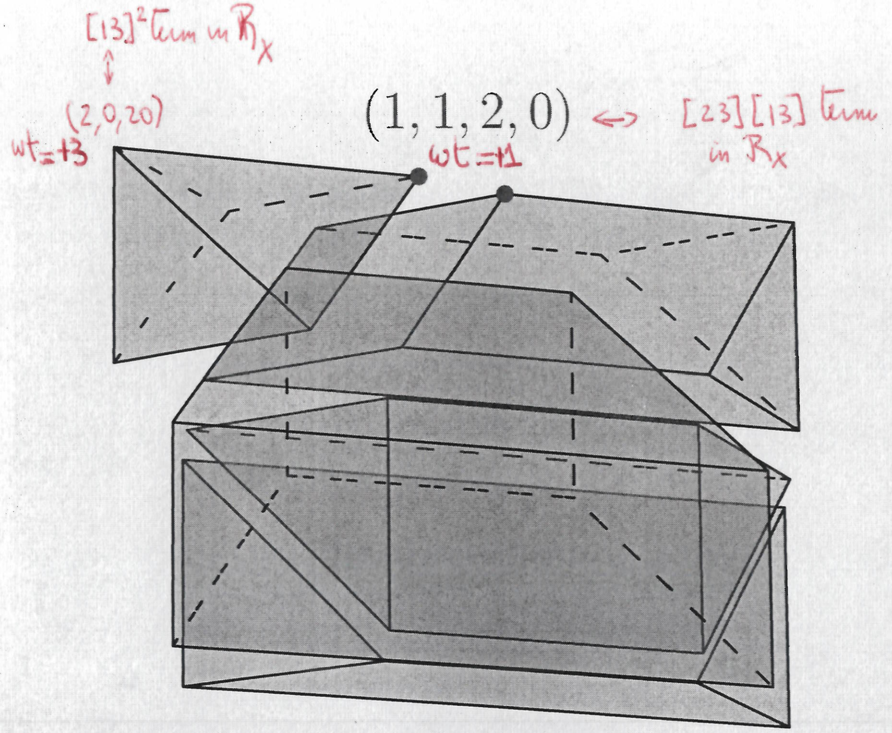
Orthant-shooting

[Example by A.Fink]

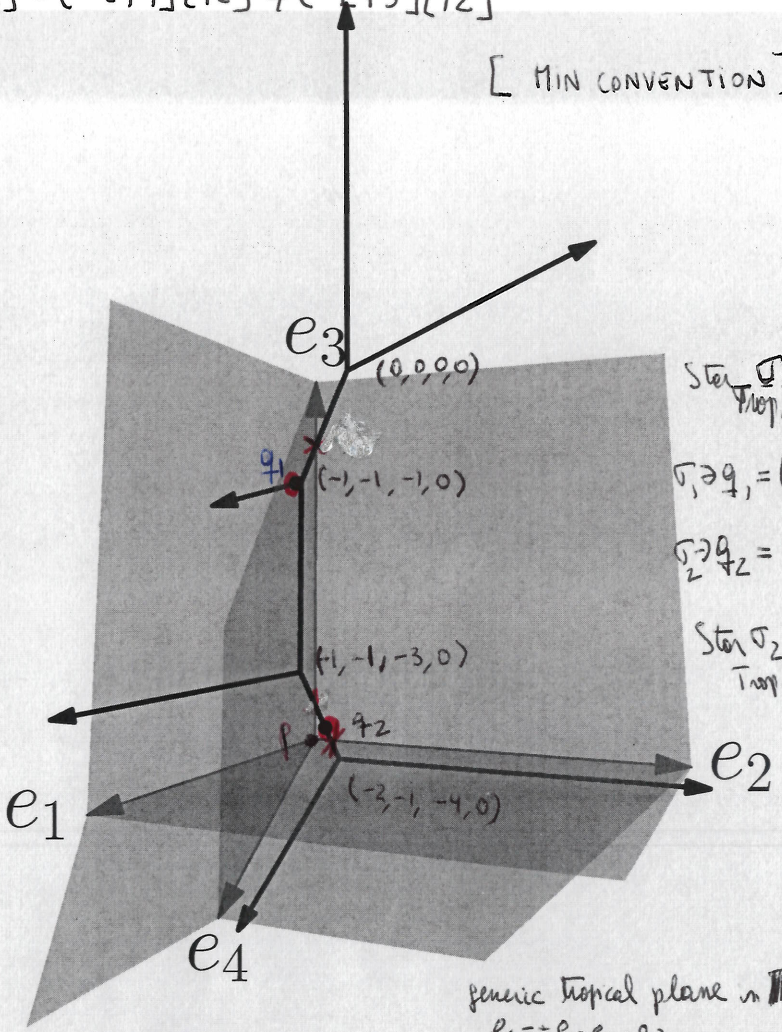
$X = V(tx - y + z - t^3 w \,;\, yz + tz^2 + t^2 yw - zw + (t^3 - t^7) w^2)$ conic in $\mathbb{P}^3$ (words $[x:y:z:w]$)

$$R_X = (2t^7+t^6+t^5-t^3)[34][24] + (t^7+t^5-t^3)[24]^2 + (2t^4+t^3+t^2-1)[34][23] + (-t^3+t^2-1)[24][23] + (-t-1)[23]^2 + (2t^8-t^4)[34]$$
$$+ (2t^8+t^6-2t^4)[24][14] + (t^9-t^5)[14]^2 + (2t^5-t)[34][13] + (-t^4+t^3-2t)[24][13] + \underline{(-2t^2-t)[23][13]} + (-t^2)[14][13]$$
$$-t^3[13]^2 + (-t^4-2t^3+t)[34][12] - t^3[24][12] + t[23][12] - t^4[14][12] + t^2[13][12]$$

Examples



[MIN CONVENTION]



$\mathrm{Star}_{\mathrm{Trop}\,X}\, \sigma_1 \cap \mathbb{Z}^3 = \mathbb{Z}\begin{bmatrix}-1\\-1\\0\end{bmatrix}$

$\sigma_1 \ni q_1 = (-\frac{5}{2}, -\frac{5}{2}, -1, 0)$

$\sigma_2 \ni q_2 = (-\frac{7}{2}, -1, -\frac{11}{2}, 0)$

$\mathrm{Star}_{\mathrm{Trop}\,X}\, \sigma_2 \cap \mathbb{Z}^3 = \mathbb{Z}\begin{bmatrix}-1\\0\\-1\end{bmatrix}$

$wt_{[1,3]} = wt(q_1) = 1 \quad \begin{vmatrix}-1 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{vmatrix} = 1$

$wt_{[2,3]} = wt(q_2) = 1 \quad \begin{vmatrix}-1 & 0 & 0\\ 0 & 0 & 1\\ -1 & 1 & 0\end{vmatrix} = 1$

$\Rightarrow R_X = [13][23] \rightsquigarrow$ vertex $= (1,1,2,0)$

generic tropical plane in $\mathbb{R}^4/\mathbb{R}\cdot 1$
$e_4 = -e_1 - e_2 - e_3$

$P = (-\frac{7}{2}, -\frac{5}{2}, -\frac{11}{2}, 0)$