Lecture XXIV: Tropical Toric Varieties

GOAL: Introduce Toric varieties & their Tropicalizations, focusing on examples.

§1 Tropicalizing Toric varieties:

Recall: $X \subset (K^\times)^n$, $\overline{K} = K$ val non trivial

$\operatorname{Trop}(X) = \text{closure}(-\operatorname{val}(X(K))) = \text{closure}(\{(-\operatorname{val}(y_1), \ldots, -\operatorname{val}(y_n)) \;\; y=(y_1,\ldots,y_n) \in X\}$ in $\mathbb{R}^n$

(I). If $X \subseteq \mathbb{A}^n \rightsquigarrow -\operatorname{val}(0) = -\infty \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$

- Put a Topology on $\overline{\mathbb{R}}$: $\bullet \cdots\cdots$ ——— $\cdots$

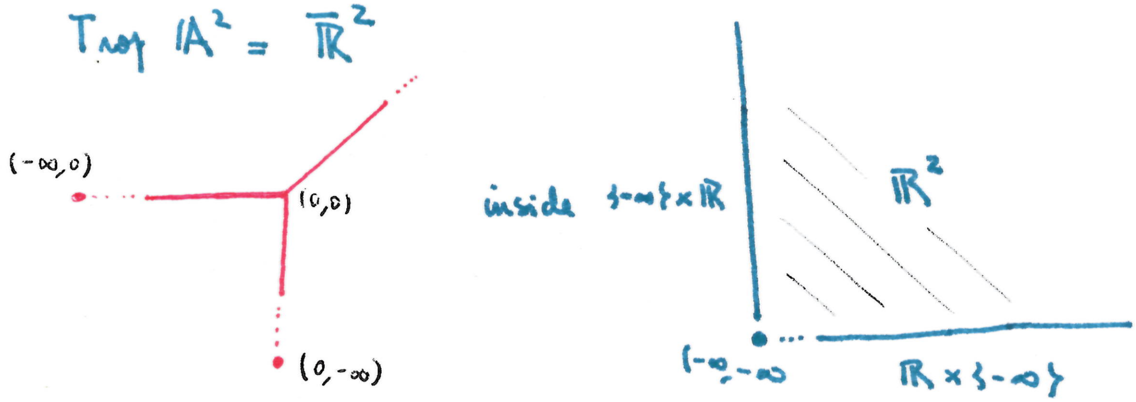
  Basis: (a,b), $[-\infty, a) = \{x \in \mathbb{R} : x < a\} \cup \{-\infty\}$ $\quad a, b \in \mathbb{R}$, $a < b$.

$\Rightarrow$ Proceed verbatim with what we did in Lectures V, VI & XI!

$$\operatorname{Trop}(X) = \text{closure}(\{(-\operatorname{val}(y_1), \ldots, -\operatorname{val}(y_n)) : y = (y_1, \ldots, y_n) \in X\}) \subseteq \overline{\mathbb{R}}^n. \quad (*)$$

Example $X = V(x+y+1) \subseteq \mathbb{A}^2$

$\operatorname{Trop} \mathbb{A}^2 = \overline{\mathbb{R}}^2$



- The Fundamental Theorem still holds!

Example: $\operatorname{Trop}(V(x+y+1)) = \{w \in \overline{\mathbb{R}}^2 : \max \text{ in } x \oplus y \oplus 0 = \max\{x, y, 0\}$ is attained $\geq$ twice $\}$

FFTG: $\operatorname{Trop}(X) = \bigcap\limits_{f \in I(X)} \mathcal{T}(V(f)) \subseteq \overline{\mathbb{R}}^n$ agrees with $(*)$

Note: The characterization in terms of initial degenerations needs more work: we need to stratify $\mathbb{A}^n$ into Tori $\{0\} \times (K^\times)^s$ $\quad X_J = \{X \cap \{x_i = 0 \Leftrightarrow i \in J\} \quad J \subseteq [n]\}$, & work with $\operatorname{in}_{w_{J^c}}(I(X_J))$ $\quad w_{J^c}$ = projection of w away from J.

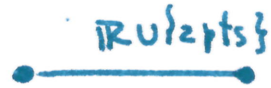
$$\cap\!\!\mid$$

$$\Bbbk[x_i^{\pm} : i \notin J] \qquad (\Bbbk = \mathcal{O}_v / m_v)$$

(II) $\mathbb{P}^n$ also works! 2 interpretations → affine cover w/ favorite coordinates in $\mathbb{A}^n$
→ $\mathbb{A}^{n+1} \setminus \{0\} / K^\times$

• **1st Perspective**:

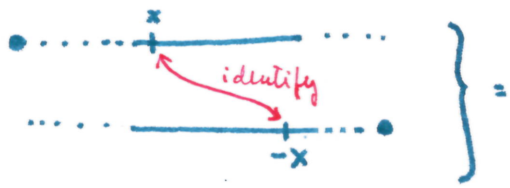
Eg: $\mathrm{Trop}(\mathbb{P}^1)$



$\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^1$ glued along $K^\times$ by $x \longrightarrow x^{-1}$ — tropicalize monomial map

So $\mathrm{Trop}(\mathbb{P}^1) = \mathrm{Trop}\,\mathbb{A}^1 \cup \mathrm{Trop}\,\mathbb{A}^1$ glued along $\mathbb{R} = \mathrm{Trop}(K^\times)$ by $x \longrightarrow -x$

$= \overline{\mathbb{R}} \cup \overline{-\mathbb{R}}$



$=$ homeomorphic to a line segment by our choice of topology

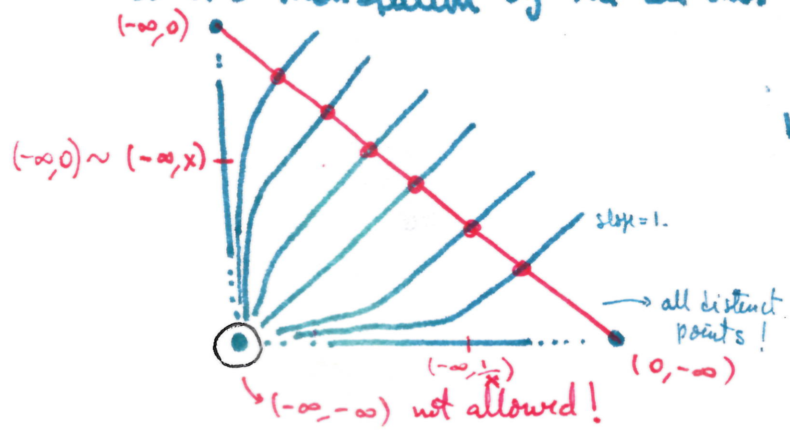
$\cong [0,1]$

• **2nd Perspective**:

Eg: $\mathbb{P}^1 = \mathbb{A}^2 \setminus \{(0,0)\} / K^\times \rightsquigarrow \mathrm{Trop}\,\mathbb{P}^1 = \left(\mathrm{Trop}\,\mathbb{A}^2 \setminus \{(-\infty,-\infty)\}\right) / \mathbb{R}$

Here: action of $\mathbb{R}$ is $\lambda \circledast (a,b) = (\lambda \odot a, \lambda \odot b) = (\lambda + a, \lambda + b)$
$= (a,b) + \lambda(1,1)$

so it's translation by the 'all-ones' vector.



• On vertical & horizontal segments which bound $\mathrm{Trop}\,\mathbb{A}^2$: all points are identified with 1 point, $(-\infty,0)$ & $(0,-\infty)$ respectively.

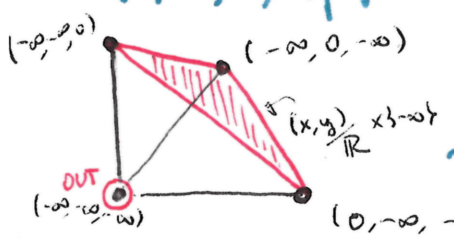
• Points in the interior of $\mathrm{Trop}\,\mathbb{A}^2$ are identified with slope one lines through ~~them~~ (translate by $(1,1)$).

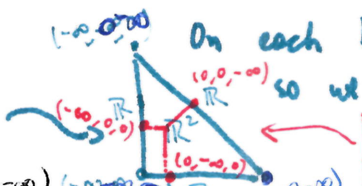
Eg 2: $\mathrm{Trop}\,\mathbb{P}^2$

View 1: $\mathbb{C}[x_0,x_1,x_2] \rightsquigarrow$ 3 charts $\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right), \left(\frac{x_0}{x_1}, \frac{x_2}{x_1}\right)\ \left(\frac{x_0}{x_2}, \frac{x_1}{x_2}\right)$

glue by Tropicalizing $(s,t) \longmapsto (s^{-1}, ts^{-1})$, etc.
monomial transition maps!

→ View 2: A point in $\mathrm{Trop}(\mathbb{P}^n)$ is an equivalence class (up to adding $\lambda(1,\dots,1)$ of pts in $\overline{\mathbb{R}}^{n+1} \setminus \{(-\infty,\dots,-\infty)\}$





On each boundary, we have $\mathbb{P}^1, \mathbb{P}^2$ (tropicalized) so we know how to identify things!

$\boxed{\text{To get } \mathrm{Trop}(X) \subseteq \mathrm{Trop}\,\mathbb{P}^n \text{: glue all } \mathrm{Trop}(X \cap \mathbb{A}^n) \text{ with fixed coord charts with tropical monomial maps}}$

§2 Intro to Toric Varieties & their tropicalization

Definition: A normal toric variety is a normal variety X containing a dense copy of $T = (K^\times)^n$ with an action of T on X extending the action of T on itself (by multiplication).

Eg: $\mathbb{A}^n \supset T = \{(x_1, \dots x_n) \mid x_i \neq 0 \ \forall i\}$

$\mathbb{P}^n \supseteq T = \{(x_0 : \dots : x_n) \mid x_i \neq 0 \ \forall i\} / K^\times \simeq (K^\times)^n$ (non-canonically, can choose 1 coord to be 1)

X (n.) Toric Variety $\Rightarrow$ X is a union of torus orbits (of dim d_i) $X = \bigsqcup_i (K^\times)^{d_i}$

$\rightsquigarrow \operatorname{Trop}(X) = \bigsqcup_i \operatorname{Trop}(K^\times)^{d_i} = \bigsqcup_i \mathbb{R}^{d_i}$

$\rightsquigarrow$ Need a topology to glue these (Tropical) torus orbits!

Miracle: Closure in this topology commutes with tropicalizing the closure in the Zariski topology

- $\operatorname{Trop}(\mathbb{A}^n) = \overline{\mathbb{R}}^n = \operatorname{Hom}_{\text{semigp}}(\mathbb{N}^n, (\overline{\mathbb{R}}, \odot)) \subseteq \overline{\mathbb{R}}^{\mathbb{N}^n}$ with product top

  ↳ determined by values at $\begin{cases} (1, 0, \dots 0) \\ (0\ 1\ 0\ \dots\ 0) \\ \quad \vdots \\ (0 \dots 0\ 1) \end{cases}$

  $\Rightarrow$ Take the induced topology from the product topology. This is the same as the one we have already constructed.

- The semigroup perspective is the easiest to generalize (b/c connection to T.V.!)

- As schemes, we glue normal toric varieties from affine normal toric varieties

- Affine normal toric varieties correspond to rational polyhedral cones $\{x \in \mathbb{R}^n : Ax \le 0\}$, $A \in \mathbb{Q}^{d \times n}$

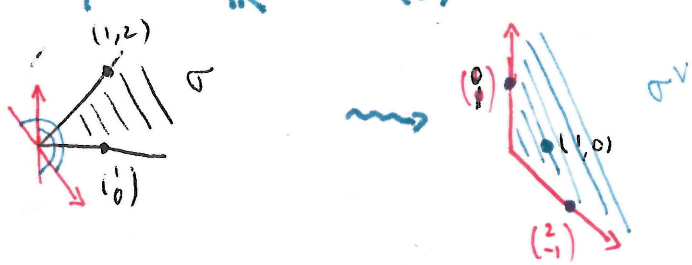
Explicitly: Fix the cocharacter lattice N of T: $N \simeq \mathbb{Z}^n$. Our rat'l polyhedral cone $\sigma \subseteq N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ has generators in N

Character lattice $M = \operatorname{Hom}(N, \mathbb{Z}^n) \simeq \mathbb{Z}^n \rightsquigarrow M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$

$\to$ think in terms of pairing of vectors!

$\sigma^\vee = \{u \in M_{\mathbb{R}} : u(v) = u \cdot v \ge 0 \ \forall v \in \sigma\} \subseteq M_{\mathbb{R}}$ (dual cone)

Example



$= \{(x,y) : \begin{matrix} x \ge 0 \\ x + 2y \ge 0 \end{matrix}\}$

• Construct the semigroup $\sigma^\vee \cap M =: S_\sigma$

Gordan's Lemma: $\sigma^\vee \cap M$ is always finitely generated as a semigroup

Example (above): $\sigma^\vee \cap \mathbb{Z}^2 = \mathbb{N}\left\langle \binom{0}{1}, \binom{1}{0}, \binom{2}{-1} \right\rangle$

Def: An affine toric variety is $U_\sigma = \mathrm{Spec}(K[\sigma^\vee \cap M])$

$= \{ \mathfrak{p}$ prime ideals in $K[\sigma^\vee \cap M] \}$

Example (above): $K[\sigma^\vee \cap M] = K[\underset{\binom{0}{1}}{y}, \underset{\binom{1}{0}}{x}, \underset{\binom{2}{-1}}{\frac{x^2}{y}}] = K[x, y, z] / (yz - x^2)$