Lecture XXV: Toric Varieties & tropical compactification

§1 Recall: Affine normal toric varieties correspond to rational polyhedral cones $\sigma = \{x \in \mathbb{R}^n : Ax \leq 0\}$, $A \in \mathbb{Q}^{d \times n}$

N = cocharacter lattice of $T = (K^\times)^n \simeq \mathbb{Z}^n$ — rat'l polyh cone $\sigma \leftrightarrow$ generators of σ in N

M = character lattice of $T \simeq \mathbb{Z}^n$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$

Given $\sigma \rightsquigarrow \sigma^\vee = \{u \in M_{\mathbb{R}} : u(v) = u \cdot v \geq 0 \ \forall v \in \sigma\} \subseteq M_{\mathbb{R}}$ (dual cone)

Semigroup $\sigma^\vee \cap M$ is a f.g. semigroup (Gordan's Lemma)

Def: An affine normal T.V. is $U_\sigma = \operatorname{Spec}(K[\sigma^\vee \cap M])$

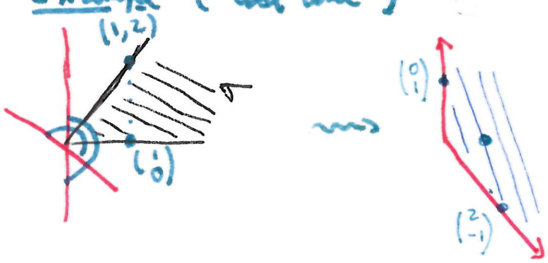
Note: The K-pts of U_σ are nothing else but $\operatorname{Hom}_{\text{semigp}}(\sigma^\vee \cap M, K)$

$\rightsquigarrow$ We can tropicalize each U_σ as $\operatorname{Trop}(U_\sigma) = \operatorname{Hom}_{\text{semigp}}(\sigma^\vee \cap M, (\overline{\mathbb{R}}, \odot))$ ↳ semigroup with mult.

Examples: (0) $\sigma = \{0\} \rightsquigarrow \sigma^\vee = \mathbb{R}^n$, $\sigma^\vee \cap M = \mathbb{Z}^n$ so $K[\mathbb{Z}^n] = K[x_1^{\pm}, \ldots, x_n^{\pm}]$ & $\operatorname{Spec} K[\mathbb{Z}^n] = T$

(1) $\sigma = \mathbb{R}^n_{\geq 0} \rightsquigarrow \sigma^\vee = \mathbb{R}^n_{\geq 0}$, $\sigma^\vee \cap M = \mathbb{N}^n$ so $K[\mathbb{N}^n] = K[x_1, \ldots, x_n]$ & $\operatorname{Spec} K[\mathbb{N}^n] = \mathbb{A}^n$.

Example (last time)



$\sigma^\vee$ — $\sigma^\vee \cap M = \mathbb{N}\langle \binom{1}{0}, \binom{0}{1}, \binom{2}{-1} \rangle \rightsquigarrow$ relation $v_3 + v_2 = 2v_1$

$$K[\sigma^\vee \cap M] = K[x, y, \tfrac{x^2}{y}] = K[x, y, z]/(yz - x^2)$$

$$\operatorname{Trop} U_\sigma = \operatorname{Hom}_{\text{sgp}}(\mathbb{N}\langle \tbinom{0}{1}, \tbinom{1}{0}, \tbinom{2}{-1} \rangle, (\overline{\mathbb{R}}, \odot)) = \{(a, b, c) : b + c = 2a\}$$

↳ · Tropicalize the defining eqn! · inherited from the semigp.

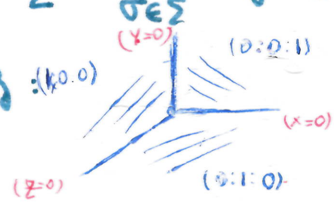
- Topology on $\operatorname{Trop} U_\sigma$ = induced by $\overline{\mathbb{R}}^{\operatorname{gen}(\sigma^\vee \cap M)}$.
- Normal toric varieties are unions of affine toric varieties, so need only tropicalize each chart & gluing maps (monomial! so we can tropicalize them nicely)
- Gluing data among cones = rational polyhedral fan Σ in $N_{\mathbb{R}}$ (not necessarily complete, i.e. $|\Sigma| = \mathbb{R}^n$)

If $\sigma \leq \tau$ cones in Σ then $U_\sigma \underset{\text{open}}{\subseteq} U_\tau$ so $\sigma = \{0\}$ ensures $T \subseteq U_\tau$ $\forall \tau$ cone of Σ.

$$X_\Sigma = \bigcup_{\sigma \in \Sigma} U_\sigma = \bigsqcup_{\sigma \in \Sigma} \mathcal{O}_\sigma \quad \text{with } \mathcal{O}_\sigma \simeq (K^\times)^{n - \dim \sigma}$$

In general: $U_\sigma \supseteq U_{\sigma \cap \tau} \subseteq U_\tau$ $\rightsquigarrow$ glue U_σ & U_τ along the open $U_{\sigma \cap \tau}$

Eg



$\mathbb{P}^2$

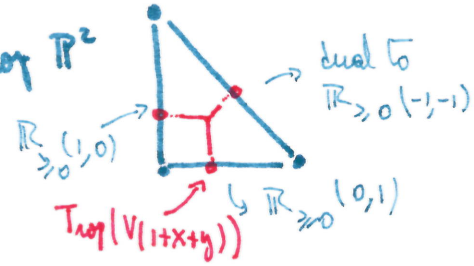
$$\overline{\mathcal{O}_\sigma} = \bigsqcup_{\tau \geq \sigma} \mathcal{O}_\tau$$

Tropicalization: $\operatorname{Trop}(X_\Sigma) = \bigsqcup_{\sigma \in \Sigma} \operatorname{Trop}(\mathcal{O}_\sigma) = \bigsqcup_{\sigma \in \Sigma} \mathbb{R}^{n - \dim \sigma} \times \{-\infty\}^{\dim \sigma}$

Trop (X_Σ) is homeomorphic to the polytope of the T.V. associated to an ample divisor

<u>Note</u>: the polytope <u>depends</u> on a choice of an ample l.b. on X_Σ but the tropical picture doesn't.

<u>Eg</u>: Trop $\mathbb{P}^2$



<u>Prop</u>: $Y \subset T \subset X_\Sigma$ & $\overline{Y}$ = closure of X in X_Σ

Then, $\text{Trop}(\overline{Y}) = \text{closure}(\text{Trop } Y) \subset \text{Trop } X_\Sigma$

§2. <u>Tropical compactification</u>

Assume K has trivial valuation

<u>Slogan</u>: Tropical geometry chooses <u>good compactifications</u> (nice boundaries!)

<u>Theorem [Tevelev]</u> Fix $Y \subseteq (K^\times)^n$ & Σ a rational polyhedral fan in $\mathbb{R}^n$

(1) The closure $\overline{Y}$ of Y in X_Σ is compact (proper) if & only if $|-\text{Trop } Y| \subseteq |\Sigma|$

[$|\Sigma|$ might be $\neq \mathbb{R}^n$, so X_Σ is not compact, $\overline{Y} \subseteq X_\Sigma$ is a <u>partial</u> compactif.]

(2) If $|\Sigma| = |-\text{Trop } Y|$, then for each cone $\sigma \in \Sigma$, $\overline{Y} \cap \mathcal{O}_\sigma$ has codimension $= \dim \sigma$ in $\overline{Y}$. (combinatorial normal crossing condition = CNC)

<u>Eg</u>: • Intersection of Y with toric divisors ($\overline{\mathcal{O}_\sigma}$ with $\sigma \in \Sigma^{(1)}$) give <u>divisors</u> on $\overline{Y}$

• Intersection with $\mathcal{O}_\sigma$ σ maximal ($\dim \sigma = \dim \overline{Y}$) gives a finite collection of points.

<u>Example 0</u>: $Y = T$, X_Σ is compact if and only if $|\Sigma| \supseteq |-\text{Trop } T| = |-\mathbb{R}^n| = \mathbb{R}^n$ (Σ complete)

<u>Example 1</u> $M_{0,n}$ Deligne-Mumford compactification of $M_{0,n}$ is a tropical compactif.

<u>Example 2</u> $Y = V(1+x+y) \subset (K^\times)^2 \rightsquigarrow X_\Sigma = X_{⅄} = \mathbb{P}^2 \smallsetminus \{(1:0:0), (0:1:0), (0:0:1)\}$

$-\text{Trop}X =$ 

$Y \simeq \mathbb{P}^1 \smallsetminus \{(0:-1), (-1:0)\}$, so $\overline{Y} \simeq \mathbb{P}^1$ & $\overline{Y} \subseteq X$ compact!

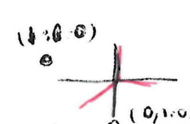
<u>Example 3</u> : $Y = V(1+2x-3y+5xy) \subset (K^\times)^2$ $\text{Trop}(Y) =$  $= -\text{Trop}(Y)$

(1) $X_\Sigma = \mathbb{P}^2 \supset \overline{Y} = V(\underline{\underline{z}}^2 + 2x\underline{\underline{z}} - 3y\underline{\underline{z}} + 5xy)$

$\overline{Y} \cap (x=0) = \{(0:1:0), (0:1:3)\} = V(z^2 - 3yz)$

$\overline{Y} \cap (y=0) = \{(1:0:0), (1:0:2)\} = V(z^2 + 2xz)$

$\overline{Y} \cap (z=0) = \{(1:0:0), (0:1:0)\} = V(5xy)$



(2) $X_{\Sigma'} = \mathbb{P}^2 \smallsetminus$ 3 fixed pts $\rightsquigarrow \overline{Y} \subset X_{\Sigma'}$ is non-compact (missing 2 pts at infinity)

(3) $X_{\Sigma''} = \mathbb{P}^1 \times \mathbb{P}^1 \supset \overline{Y} = V(x_0 y_0 + 2x_1 y_0 - 3 y_1 x_0 + 5 x_1 y_1)$ bidegree $(1,1)$

Pts $= (x_0 : x_1), (y_0 : y_1)$

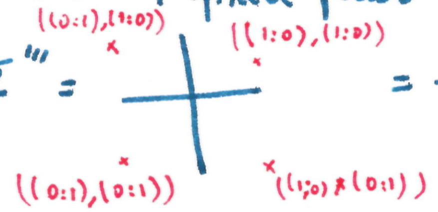
$\overline{Y} \cap (x_0 = 0) = \{((0:1),(5:-2))\} = V(2 \underbrace{x_1}_{=1} y_0 + 5 \underbrace{x_1}_{=1} y_1)$

$\overline{Y} \cap (x_1 = 0) = \{((1:0),(3:1))\} = V(y_0 - 3y_1)$

$\overline{Y} \cap (y_0 = 0) = \{((5:3),(0:1))\} = V(-3x_0 + 5x_1)$

$\overline{Y} \cap (y_1 = 0) = \{((2:-1),(1:0))\} = V(x_0 + 2x_1)$

(4) $X_{\Sigma'''} = \mathbb{P}^1 \times \mathbb{P}^1 \setminus 4$ fixed points $\rightsquigarrow \overline{Y} \subseteq X_{\Sigma'''}$ is compact!

$\Sigma''' =$ 

$= -\mathrm{Trop}(Y)$

- Theory extends to arbitrary valuations, replacing toric varieties with toric schemes!