# Lecture XXVI: Geometric Tropicalization - Divisorial Valuations

Last Time: given $Y \subset (K^*)^n$, $K$ with trivial valuation, $\Sigma = -\text{Trop}\, Y$ fan, induces

(1) $\overline{Y} \subset X_\Sigma$ compactification

(2) $\partial Y = \overline{Y} \smallsetminus Y$ is divisorial (components are divisors in $\overline{Y}$ = codim 1 subvars of $\overline{Y}$)

& for each cone $\sigma \in \Sigma$, $\overline{Y} \cap \mathcal{O}_\sigma$ has codim = dim $\sigma$ in $\overline{Y}$.

[intersection with $\mathcal{O}_\sigma$ for $\sigma \in \Sigma^{(1)}$ has codim = 1]

Q: Can we reverse the construction? ie compute Trop $Y$ from a good compactification?

A: YES $\leadsto$ Geometric Tropicalization [Hacking-Keel-Tevelev].

- Assume: val on $K$ is trivial & $Y \subset (K^*)^n$ is irreducible.

KEY idea: Characterization of Trop$(Y)$ via divisorial valuations

## §1 Divisorial Valuations:

Given $Y \subset (K^*)^n$ irred, $K[Y] = \underbrace{K[x_1^{\pm}, \ldots, x_n^{\pm}]}_{I}$ domain $\leadsto$ $K(Y) = \text{Quot}(K[Y])$ field of fractions

Defn: A variety $Y'$ is birational to $Y$ if $\exists$ $\underset{\text{open}}{Y \supset U} \xrightarrow[\text{iso}]{\sim} \underset{\text{open}}{V \subset Y'}$.

[Example: $Y' = \overline{Y} \subset X_\Sigma$]

- Pick $Y'$ birational to $Y$ satisfying (1) $Y'$ is normal

(2) $Y'$ is $\mathbb{Q}$-factorial (a multiple of every Weil divisor is Cartier)

Implication: Every codimension-1 subvariety of $Y'$ is locally defined by a single equation

By construction, the same holds for $Y$ & any open affine $\mathcal{U}$ of $Y'$.

- Prop: A prime divisor in $Y'$ determines a (divisorial) valuation on $K(Y)$, trivial on $K$.

Proof: $Y'$ normal, so any $\mathcal{U} \subseteq Y'$ affine open has $K[\mathcal{U}]$ normal.

- Pick $\mathcal{U}$ meeting $D$ & let $\mathfrak{P} \subset K[\mathcal{U}]$ prime ideal defining $D \cap \mathcal{U}$
- $\mathfrak{P}$ has codim 1 in $K[\mathcal{U}]$ & it's defined by a single eqn $= (f) = \mathfrak{P}$.

We have $K[\mathcal{U}]_{\mathfrak{P}}$ is a DVR & we write $\text{val}_D : K(Y) \longrightarrow \mathbb{Z}$ for the $(f)$-adic valuation on $K(Y) = \text{Quot}(K[\mathcal{U}]_{\mathfrak{P}})$. This is the divisorial valn induced by $D$.

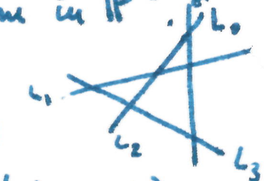
- $\text{val}_D$ is trivial on $K$ by construction. $\square$

**KEY**: $\mathrm{val}_D$ is independent on $\mathcal{U}$ & on the birational model since $K(Y')=K(Y)$ & divisors on $Y'$ give divisorial valuations on $Y$.

• $(f)$-adic valuation: write $g = f^n h$ for $(f:h)=1$ with $n\in\mathbb{Z}$. Think of $n$ as the order of $(f)$ as a zero of $g$ (if $n\geq 0$) or as neg order of $(f)$ as a pole of $g$ (if $n<0$)

Example $Y = V(1+x_1+x_2+x_3) \subseteq (K^*)^3$, $Y' = \overline{Y} = V(x_0+x_1+x_2+x_3) \subseteq \mathbb{P}^3$.

$\partial Y$ = 4 lines $= L_0, L_1, L_2, L_3$, $L_i = \overline{Y} \cap \{x_i=0\}$ = line in $\mathbb{P}^2$.




Take $\mathcal{U} = (x_0\neq 0)$ ⇝ coordinates $\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}$ $\left(y_i := \frac{x_i}{x_0}\right)$

$K[\mathcal{U}] = K[y_1,y_2,y_3]/(1+y_1+y_2+y_3)$

Equation for $L_1$ in $\mathcal{U} = (y_1=0)$

Write any $g$ in $K(\mathcal{U})$ as $g = y_1^n h$ where $(h,y_1)=1$. (factor num & denom of $g$)

Then $\mathrm{val}_{L_1}(g) = n$. $K(\mathcal{U})_{\langle y_1\rangle}$ ↳ regular local ring of dim $=1$ so DVR

**Def**: Fix a normal $\mathbb{Q}$-factorial compactification of $Y\subset (K^*)^n$; $M = \mathrm{Hom}((K^*)^n, K^*) = \mathbb{Z}\langle x_1,\ldots,x_n\rangle$

To a divisorial valuation $\mathrm{val}_D$ on $K(Y)|K$ we associate a vector
$$[\mathrm{val}_D] = (\mathrm{val}_D(x_1),\ldots,\mathrm{val}_D(x_n)) \in M^\vee = N,$$
(This comes from a choice of basis for $M$, in general $\mathrm{val}_D : M_{\mathbb{R}} \to \mathbb{R}$)

Example (cont.) $[\mathrm{val}\, L_0] = [-1:-1:-1]$

$M = \mathbb{Z}\langle \frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\rangle$ characters of $(K^*)^3$.

$[\mathrm{val}\, L_1] = [1:0:0]$

$[\mathrm{val}\, L_2] = [0:1:0]$

$[\mathrm{val}\, L_3] = [0:0:1]$

**Aside**: We'll use a similar construction for any valuation on $Y$ ⇝ Berkovich spaces!

**Prop**: Assume $Y\subset (K^*)^n$ irred, valn on $K$ trivial. Then:
$$\mathrm{Trop}(Y) = \text{closure}\left(\{-c[\mathrm{val}_D] : c\in\mathbb{Q}_{\geq 0},\ \mathrm{val}_D \text{ is a divisorial valn on } K(Y)|K\}\right) \text{ in } \mathbb{R}^n$$

Proof of $\supseteq$: Assume $Y$ is defined by a prime ideal $I \subset K[x_1^{\pm},\ldots,x_n^{\pm}]$ & let $\mathrm{val}_D$ be a divisorial valuation on $K(Y)$. Write $\omega = [\mathrm{val}_D] = (\mathrm{val}(x_1),\ldots,\mathrm{val}(x_n))$

Want to show: $-\omega \in \mathrm{Trop}(Y)$. ($\mathrm{Trop}(Y)$ is a fan, so this is enough)

Pick $f = \sum c_u x^u \in I$, then $f = 0$ in $K(Y)$

Then $-\infty = -\mathrm{val}_D(0) = -\mathrm{val}_D(f)$

Compare with $-\infty \leq \max\limits_{\substack{u \\ c_u \neq 0}} \{-\mathrm{val}_D(c_u x^u)\} \underset{\mathrm{val}|_K = 0}{=} \max\limits_u \{-\mathrm{val}_D(x^u)\} = \max\limits_u (-w \cdot u) = \mathrm{trop}(f)(w)$

If max is attained only once, then $-\mathrm{val}_D(\sum c_u x^u) = \max\limits_u -\mathrm{val}_D(c_u x^u) \neq -\infty$ Contr!

We conclude $(-w)$ is a zero of $\mathrm{trop}(f)$, so $-w \in \mathcal{T}(V(f))$ for any $f$ in $I$.

Conclusion: $-w \in \mathrm{Trop}(Y)$.

- For $\supseteq$: Need to show that any $-w \in \mathrm{Trop}\,Y \cap \mathbb{Q}^n$ primitive we can find $c \in \mathbb{Q}_{>0}$ & $D$ a divisor on $Y$ s.t. $-w = c[\mathrm{val}_D]$

($D$ will be a Cartier divisor on $Y'$ = normalization of $\overline{Y} \subset X_\Sigma$ where $\Sigma = \mathbb{R}_{\geq 0}(w)$)
$\nu: Y' \to \overline{Y}$ $\qquad X_\Sigma \cong \mathbb{A}^1 \times (K^*)^{n-1}$

$\overline{Y}$ meets $D_1$ = torus-invariant divisor defined by $x^w$.

$D$ = any irreducible component of $\nu^{-1}(\overline{Y} \cap D_1)$

($\nu^*(x^w)$ vanishes on $D$ so $\mathrm{val}_D(\nu^*(x^w)) > 0$.

On the complementary characters $x^\eta$: $\nu^*(x^\eta) = 0$.
$M \cap \langle w \rangle^\perp = \langle \eta_1, \dots, \eta_{n-1} \rangle$

Claim: $w = c[\mathrm{val}_D]$ for $c = \dfrac{1}{\mathrm{val}_D(\nu^*(x^w))}$

(Think of $w = e_1$, complementary characters $x_2, \dots, x_n$ (so $x^w = x_1$)) $\square$

## §2 Geometric Tropicalization

Given $\partial Y = \overline{Y} \setminus Y$ divisorial boundary with components $\partial\overline{Y} = D_1 \cup \dots \cup D_s$ ($D_i$ irred divisor on $\overline{Y}$).

Desirable property: simple normal crossings ($\simeq$ arrangement to some coordinate hyperplanes)

From Tropical Compactification get combinatorial normal crossings ($r$ components meet in codimension $r$)

Eg: (in dim 2)

SNC
CNC

not SNC
not CNC

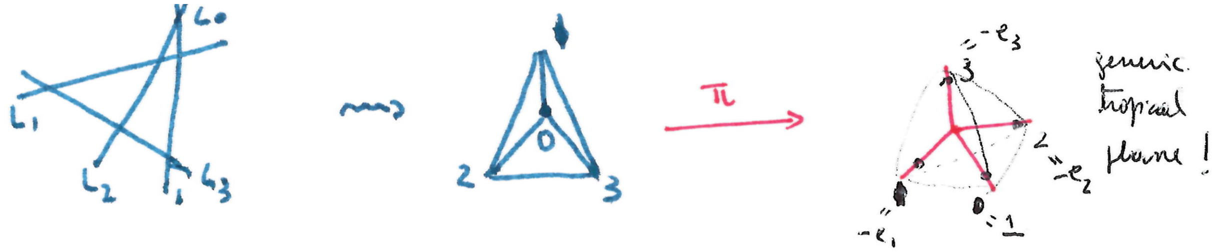
not SNC
BUT CNC.

**Definition** Assume $\partial Y$ divisorial.

The boundary complex $\Delta(\overline{Y})$ of the pair $(\overline{Y}, \partial\overline{Y})$ is the simplicial complex with
- vertices on $\{1, \dots, s\}$ ($i \longleftrightarrow D_i$)
- simplex $\sigma = \{i_1, \dots, i_r\}$ whenever $D_{i_1} \cap \dots \cap D_{i_r} \neq \emptyset$)

Alternative: One simplex $\sigma$ per component of $D_{i_1} \cap \dots \cap D_{i_r}$ ⇝ $\Delta$-complex. [Bayne]

Example (cont.)




Thm [HKT] Given $\overline{Y}$ normal, $\mathbb{Q}$-factorial, $\partial\overline{Y}$ divisorial & let $\pi : \Delta(\overline{Y}) \longrightarrow \mathbb{R}^n$, $v_i \longmapsto [\mathrm{val}_{D_i}]$ extended linearly on simplices. Then, the cone over the image of $\pi$ contains $-\mathrm{Trop}\, Y$.

If $(\overline{Y}, \partial\overline{Y})$ is smooth & SNC (or char 0 & CNC) we have equality. (as sets, not as fans!)

Thm [C] In the nice setting we have a formula for computing tropical multiplicities from intersection numbers between various $D_i$'s.