Lecture XXVII: Berkovich Analytic Spaces from the tropical viewpoint I

Overview: ① Tropical Geometry is a coordinatewise dependent shadow of $Y \subset (K^*)^n$
($\& \; Y \subset X_\Sigma$ T.V.)

⇝ Q1: What happens when we change the embeddings equivariantly?

Q2: Can we find a topological space containing ALL tropicalizations?

A: YES! Berkovich non-Archimedean analytification: Y^{an}.

② How to effectively decide which embeddings best reflect the geometry of Y^{an}?

§1 Setup

Fix (K, val) rank-1 valuation $val: K^\times \longrightarrow \mathbb{R}$. Then val induces a topology on K via a non-Archimedean absolute value $|\;|: K \longrightarrow \mathbb{R}_{\geq 0}$ $\quad |x| = e^{-val(x)}$.

Properties:
- $|a| = 0 \iff a = 0$ (norm: no kernel)
- $|a \cdot b| = |a|\,|b|$ (multiplicative)
- $|a+b| = e^{-val(a+b)} \leq e^{\max\{-val(a), -val(b)\}} = \max\{|a|, |b|\}$
  $=$ if $|a| \neq |b|$.

non-Archimedean
Δ-inequality
ULTRAMETRIC

Non-Archimedean: $|\underbrace{1+\cdots+1}_{n \text{ times}}| \leq 1 \quad \forall n$.

TOPOLOGY: Basis $\underset{x \in K,\, r>0}{} \; B_0(x,r) = \{y : |x-y| < r\}$ $\quad (B(x,r) = \{y : |x-y| \leq r\})$

Prop: Open balls are closed!

Pf/ $|x-z| \geq r \Rightarrow$ Pick any $0 < \varepsilon < r$ & show $B_0(z,\varepsilon) \subset K \smallsetminus B_0(x,r)$

Why? $|x-y| = |\underbrace{(x-z)}_{|\;| \geq r} + \underbrace{(z-y)}_{< \varepsilon < r}| = \max\{|x-z|, |z-y|\} = |x-z| \geq r$
if $y \in B_0(z,\varepsilon)$ □

Equivalently: Every pt y in $B(x,r)$ gives $B(x,r) = B(y,r)$

Pf/ Enough to show $(\subseteq)$ $|y-z| = |y-x+x-z| \leq r$ for any $z \in B(x,r)$ □

Conclusion: Two discs are either disjoint or the same! $\Rightarrow$ Topology is totally disconnected!

⇝ analysis on these spaces breaks down!

Solns: (1) Rigid analytic geometry (Tate) ⇝ work with Grothendieck topology

(2) Berkovich: Add points to K (resp. to any scheme of finite type$/K$) to fix the nasty topology (Zariski + non-Arch top on K)

For most applications & topological properties, we want K to be complete w.r.t abs. value on K.

- If not, take $\hat{K}$ completion (from metric on K) & give it the extended

$|\ |: \hat{K} \longrightarrow \mathbb{R}_{\geq 0}$ $\quad |\lim_\alpha x_\alpha| := \lim_\alpha |x_\alpha|$ for any Cauchy seq $(x_\alpha)_\alpha$

(show $||x_\alpha| - |x_\beta|| \leq |x_\alpha - x_\beta| \xrightarrow[\alpha,\beta\to 0]{} 0$ because (x_α) is Cauchy, $\mathbb{R}_{\geq 0}$ is complete).

- Show $(\hat{K}, |\ |)$ is non-Arch. complete abs. value. (non val: $\hat{K}^\times \to \mathbb{R}$ $\mathrm{val}(x) = -\log|x|_{\hat{K}}$ is the ! valn on $\hat{K}$ extending val on K)

Remark 1: $\hat{\overline{K}}$ is both complete & alg closed if $K = \overline{K}$.

Examples: (1) Any K with trivial valn

(2) $K = \widehat{\overline{\mathbb{C}((t))}} \subsetneq \mathbb{C}((\mathbb{R}))$ = generalized Puiseux series with t-valn

(3) $K = \mathbb{C}_p := \widehat{\overline{\mathbb{Q}}_p}$ with p-adic valn. $|\frac{a}{b}|_p = p^{-r}$ if $\frac{a}{b} = p^r \frac{c}{d}$ $p \nmid c, d$.

(2 points are closed p-adically if their difference is divisible by a large power of p)

Remark 2: $\Gamma_{\mathrm{val\ on\ } K} = \Gamma_{\mathrm{val\ on\ } \hat{K}}$.

§2 Berkovich Analytification:

Borrow terminology from valn: $K^\circ = \{f \in K : |f| \leq 1\} = \mathcal{O}_{\mathrm{val}}$, $\quad \tilde{K} = K^\circ / K^{\circ\circ}$ res. field

$K^{\circ\circ} = \{f \in K : |f| < 1\} = \mathfrak{m}_{\mathrm{val}}$

X = K-scheme of finite type $\underset{\boxed{\text{GAGA functor}}}{\rightsquigarrow}$ X^{an} = Top space + sheaf of analytic functions

Focus on this for tropicalization purposes

- Build X^{an} by gluing $(\mathrm{Spec}\, A)^{an}$ for A f.g K-algebra, say $A = K[x_1 \cdots x_n]/I$.

Def: $(\mathrm{Spec}\, A)^{an} = \{\|\ \|: A \longrightarrow \mathbb{R}_{\geq 0}$ multiplicative seminorms on A extending $|\ |_K$ & satisfying non-Arch Δ-inequality$\}$

seminorm: $\|a\| = 0 \not\Rightarrow a = 0$.

- In particular: $\|f\| = \|f + I\|$, $\|f\| = 0$ for all $f \in I$, further $\forall f \in \sqrt{I}$
- Can identify $\|\ \| \in (\mathrm{Spec}\, A)^{an}$ with $\|\ \| \in (\mathrm{Spec}\, K[x_1 \ldots x_n])^{an}$ with $\mathrm{Ker}\, \|\ \| \supseteq I$.

TOPOLOGY: Weakest such that $ev_f : \|\ \| \longmapsto \|f\|$ is continuous $\underline{\underline{\forall}}\, f \in A$.

(evaluation maps)

$\rightsquigarrow (\mathrm{Spec}\, A)^{an} \subset \mathbb{R}_{\geq 0}^{A}$ with product topology.

In general: (original setting) A Banach ring with norm $\|\ \|$ $\rightsquigarrow \mathcal{M}(A) := \{ |\ | : A \longrightarrow \mathbb{R}_{\geq 0}$ satisfying (i) $|1| \leq 1$, (ii) $|fg| \leq |f|\,|g|$, (iii) $|f| \leq C\|f\|$ ($|\ |$ bounded) $\forall f\}$

- If $\|\ \|$ multiplicative, can take $C = 1$

Then $\mathcal{M}(A) \subseteq \prod_{f \in A} [0, \|f\|]$ compact | $\mathcal{M}(A) \subseteq \bigcup_{C>0} [0, C\|f\|]$ prod top

HARD: Show $\mathcal{M}(A) \neq \{|\ |\}$

Note: (1) $X(K) \hookrightarrow X^{an}$

$\underset{K\text{-alg map}}{\varphi : A \to K} \longmapsto (\| \, \|_\varphi : f \mapsto |\varphi(f)|)$ $\quad (X = \operatorname{Spec} A)$

(2) Compatible w/ valued field extensions $(L, val_L) \mid (K, val_K)$, ie $val_L|_K = val_K$
$\rightsquigarrow \exists \, | \, |_L$ extending $| \, |_K$

$$\begin{tikzcd} X(L) \arrow[r] & X^{an} \\ X(K) \arrow[u] \arrow[ur, hook] \arrow[r, phantom, "\circlearrowleft"] & \end{tikzcd}$$

(analytif for X as K-scheme)

Do it for $X = \operatorname{Spec} A$
$X_L = \operatorname{Spec} A \otimes L$
& glue.

Theorem [Berkovich '90]

(1) X^{an} is locally compact, locally path connected

(2) X connected $\iff$ X^{an} is path connected

$X|K$ separated $\iff$ X^{an} is Hausdorff $\quad$ (—:— not separated)

$X|K$ proper $\iff$ X^{an} is compact

(3) If $| \, |_K$ is non-trivial, then $X(\bar{K})$ is dense in X^{an}.

Example $X = \mathbb{A}^1$, $| \, |_K$ trivial $\quad 0 \neq f = \sum_{i=m}^{N} c_i x^i \quad \rightsquigarrow \|f\| \leq \max_{c_i \neq 0} \{ \overset{1}{\overbrace{|c_i|}} \|x^i\| \}$ (*)

$\mathbb{A}^1 = \operatorname{Spec} K[x]$ $\qquad c_m, c_N \neq 0$ $\qquad = \max_{c_i \neq 0} \{ \|x\|^i \}$

& = if all $\|x\|^i$ are $\neq$!

Q: Can we determine $\|f\|$ from $\|x\|$? A: Not always! $\quad \|0\| = 0$ ✓

For $f \neq 0$

$$\|f\| = \begin{cases} \|x\|^N & \|x\| > 1 \\ \|x\|^m & \|x\| < 1 \\ ? & \|x\| = 1 \end{cases}$$

As $\|x\| \to 1$ in both cases $\| \, \|$ becomes the trivial norm η

$\|f\| \to 1 \quad \forall f \neq 0$ $\qquad \eta(f) = \begin{cases} 0 & f = 0 \\ 1 & f \neq 0 \end{cases}$

$\|f\| \leq 1 \quad \forall f$ $\rightsquigarrow$ $\mathfrak{P} = \{ f : \|f\| < 1 \}$ is prime ideal, $\mathfrak{P} \subsetneq K[x]$ $(1 \notin \mathfrak{P})$

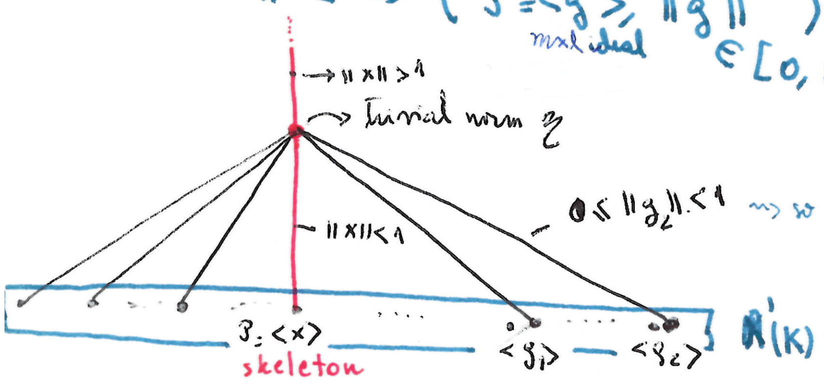
by (*) $\qquad \Rightarrow$ either $\mathfrak{P} = 0$ or maxl ideal

. If $\mathfrak{P} = \{0\}$, then $\| \, \|$ is trivial

. If $\mathfrak{P} \neq \{0\}$, write $\mathfrak{P} = \langle g \rangle$ because $K[x]$ is PID, g irred

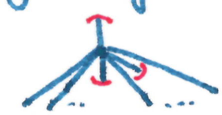
Then, write $0 \neq f = g^s h$ for $(g, h) = 1$, so $\|h\| = 1$. Then $\|f\| = \|g\|^s$.

So $\|f\| \longleftrightarrow (\underset{\text{maxl ideal}}{\mathfrak{P} = \langle g \rangle}, \|g\| \in [0, 1))$



$\xrightarrow{\log}$ $\overline{\mathbb{R}} = \operatorname{Trop}(\mathbb{A}^1)$

Properties:

- Topology on each open segment (branch $\neq \dot{I}$) as $\subseteq \mathbb{R}$.
- nbhd of η = All branches except finitely many are replaced by open segments (E_g )

  Why? Pick any $g \neq 0$, irred, not a unit.

  $ev_g : \| \ \| \longmapsto \|g\| = 1$ on all $\| \ \|$ in branches $\neq$ Branch from g & also Branch $\|x\|>1$.

  $ev_g^{-1}((1-\varepsilon, 1+\varepsilon))$ contains all branches that are not $\langle g \rangle$-branch.
- Only 1 branch pt = η (with infinitely many branches)
- Leaves = K-points
- $(\mathbb{A}^1)^{an}$ contracts to η.