Lecture XXVIII: Berkovich Analytic Spaces II

Last Time: $(K, |\;|_K)$ non-Arch. field ($|\;|_K$ mult. norm, non-Arch Δ-ineq), K complete

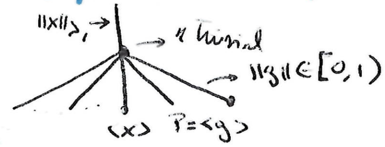
X: K scheme of f. type, say $X = \operatorname{Spec} A$.

$X^{an} = \{ \|\;\| : A \to \mathbb{R}_{\geq 0}$ mult seminorm extending $|\;|_K$ & satisfying (non Arch) Δ-ineq $\}$

(*) only need to check $\|f+g\| \leq \|f\| + \|g\|$

(In general: X^{an} glued from $(\operatorname{Spec} A)^{an}$ w/ gluing maps φ_{ij}^{an}, φ_{ij}: gluing for X)

Ex: $(\mathbb{A}^1)^{an}$ w/ $|\;|$ trivial



(*) $\|(f+g)\|^n = \left|\sum_i \binom{n}{i} f^i g^{n-i}\right|$

$\leq \sum \|\binom{n}{i}\| \, \|f\|^i \|g\|^{n-i}$

$\leq C\,(n+1) \max\{\|f\|, \|g\|\}^n$ (K-non Arch, $C = \|1\|$)

& take $\sqrt[n]{\;}$ & $n \to \infty$

TODAY: arbitrary valuation

§1 $(\mathbb{A}^1)^{an}$ with K non-trivial valued, $\overline{K} = K$:

Example: $K = \mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$ ($|\frac{a}{b}|_p = |p^r \frac{a'}{b'}| = p^{-r}$ $p \nmid a', b'$.)

GOAL: Construct seminorms on $K[T]$

Def: $D(a,r) = \{ z \in K : |a - z| \leq r \} \subseteq K$ $\quad a \in K, r \geq 0$ discs.

$\rightsquigarrow |\;|_{D(a,r)} : K[T] \longrightarrow \mathbb{R}_{\geq 0}$ $\quad |f|_{D(a,r)} = \sup_{z \in D(a,r)} |f(z)|$ (sup-norm)

Prop $|\;|_{D(a,r)}$ is multiplicative & $|f+g|_{D(a,r)} \leq |f|_{D(a,r)} + |g|_{D(a,r)}$.

Pf/ Use Gauss' lemma to factor $f = c \prod_{i=1}^{N} (T - \alpha_i)$

Assume $c = 1$ since $|f| = |c| \, |\frac{f}{c}|$.

- If $\alpha_i \notin D(a,r)$ $\quad |z - \alpha_i| = |\underbrace{z-a}_{\leq r} + \underbrace{a - \alpha_i}_{> r}| = |a - \alpha_i|$ $\quad \underline{\forall}\, z \in D(a,r)$
- If $\alpha_i \in D(a,r) \underset{\text{Last Time}}{=} D(\alpha_i, r)$ $\Rightarrow |z - \alpha_i|$ is maximal over any pt in the boundary of the disc & value $= r$.

$\Rightarrow |f|_{D(a,r)} = r^{\#\{\alpha_i \in D(a,r)\}} \prod_{\alpha_i \notin D(a,r)} |a - \alpha_i|$ $\quad$ The formula is multiplicative □

Corollary: $|\;|_{D(a,r)} \in (\mathbb{A}^1_K)^{an}$.

Remarks: (1) If $r \notin |K^\times|$, then the sup will not be achieved if f has a root $\in D(a,r)$.

(2) View $K \hookrightarrow (\mathbb{A}^1)^{an}$ as $a \longmapsto |\;|_{D(a,0)}$ = evaluation at a of $f \in K[T]$

(3) If $r > 0$: $|\;|_{D(a,r)}$ is a NORM b/c $|T-b|_{D(a,r)} = \begin{cases} r \neq 0 & \text{if } b \in D(a,r) \\ |b-a| \neq 0 & \text{if } b \notin D(a,r) \end{cases}$

(4) $|\;|_{D(a,r)}$ are all distinct.

Pf/ If ~~Balls~~ are disjoint: $|T-a'|_{D(a,r)} = |a-a'| > r'$ but $|T-a'|_{D(a',r')} = r'$

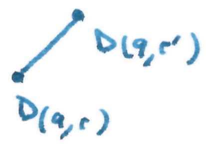
- If $B(a,r) \underset{= D(a,r')}{\subsetneq} D(a',r')$ pick $b \in B_{(a',r')} \setminus D(a,r)$ ($K = \overline{K}$, Γ_{val} is dense). Then $|T-b|_{D(a,r)} = |b-a| < r'$, $|T-b|_{D(a',r')} = r'$

Q: Path connecting $|\,|_{D(a,r)}$ to $|\,|_{D(a',r')}$? (includes $r=0$ or $r'=0$ cases!)

A: Order discs by containment.

CASE 1: $a=a'$ & $r<r'$ $\varphi:[0,1]\longrightarrow(\mathbb{A}^1)^{an}$ continuous.

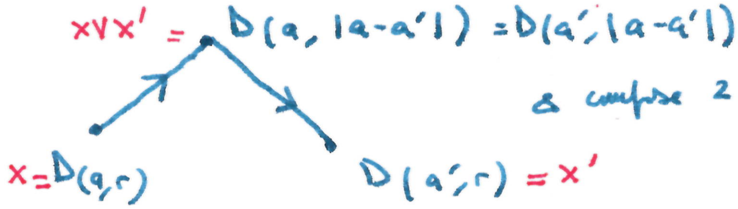
$t\longmapsto |\,|_{D(a,\,tr'+(1-t)r)}$



CASE 2: Disjoint balls:

$|a-a'|=s$

$s>r,r'$



& compose 2 paths.

• Missing pts? If K is not spherically complete ($\exists$ decreasing sequences of nested discs w/ empty intersection)

These seq. define pts in $(\mathbb{A}^1)^{an}$: Say $D(a_n,r_n)$ nested $\bigcap_n D(a_n,r_n)=\emptyset$

Then $|\,|_{\{D(a_n,r_n)\}_{n\in\mathbb{N}}}: f\longmapsto \lim_{n\to\infty}|f|_{D(a_n,r_n)}$ is a multiplicative seminorm on $K[T]$ extending $|\,|_K$. In fact, it's a norm.

Key: • For empty intersection, we have $r_i \searrow_i r>0$ (because K is complete)

• If $b\notin D(a_n,r_n)$ then $|T-b|_{D(a_s,r_s)}=|b-a_n|\ \forall s>n$.

(b/c $|b-a_s|=|\underbrace{b-a_n}_{>r_n}+\underbrace{a_s-a_n}_{\leq r_n}|=|b-a_n|\ \forall s>n$)

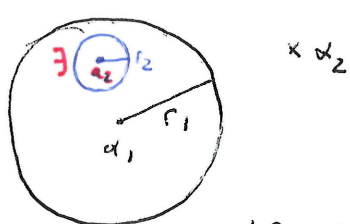
Example: $\mathbb{C}_p$ is NOT spherically complete.

STS $\mathbb{C}_p\cap D(0,1)$ is not sph complete. Write $\mathbb{Q}\cap D(0,1)=\{\alpha_j\}_{j\geq1}$ countable & dense in $D(0,1)$.

Use $\Gamma_{val_p}=\mathbb{Q}$ to construct a sequence $r_1>r_2>\dots\geq0$ in $|\mathbb{Q}|$ such that: $\alpha_1=a_1, a_2,\dots$ in $\mathbb{C}_p\cap D(0,1)$

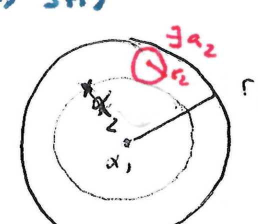
$D(a_i,r_i)\supseteq D(a_{i+1},r_{i+1})$, $\alpha_1,\dots,\alpha_s\notin D(a_{s+1},r_{s+1})$

CASE 1



$r_2=\min\{r_1,\tfrac{1}{2}|a_2-\alpha_1|\}$

CASE 2



$\exists\ a_2\in D(\alpha_1,r_1)\setminus D(\alpha_1,|\alpha_1-\alpha_2|)$

$r_2=\min\{r_1,\tfrac{1}{2}|a_2-\alpha_2|,\tfrac{1}{2}|a_2-\alpha_1|\}$

Berkovich's Classification Thm: These are all the points of $(\mathbb{A}^1)^{an} \leftrightsquigarrow$ nested sequences of discs in K

(1) Type I: $\lim_{n\to\infty} r_n = 0 \underset{K \text{ complete}}{\Longrightarrow} \bigcap_i D(a_i,r_i) = \{a\} \subseteq K \rightsquigarrow pt = |\;|_{D(a,0)}$

(2) Type II: $\lim_{n\to\infty} r_n = r \in |K^\times|$ & $\exists\, a \in \bigcap_i D(a_i,r_i) \rightsquigarrow pt = |\;|_{D(a,r)}$ (rational!)

(3) Type III: $\lim_{n\to\infty} r_n = r \notin |K^\times|$ & $\exists\, a \in \bigcap_i D(a_i,r_i) \rightsquigarrow pt = |\;|_{D(a,r)}$ (irrational!)

(4) Type IV: $\bigcap_i D(a_i,r_i) = \emptyset$ (& $\lim_{n\to\infty} r_n = r > 0$)

Remark: Two nested sequences of discs give the same seminorm iff

(1) they have the same non-empty intersection

or

(2) They are cofinal (they interlace) & both have empty intersection.

• **Partial order on $(\mathbb{A}^1)^{an}$**: $x = [\;]_x \le [\;]_y = y \iff [f]_x \le [f]_y \quad \forall f \in K[T]$.

Equivalently: If $[\;]_x = |\;|_{D(a,r)}$ & $[\;]_y = |\;|_{D(a',r')}$ then $[\;]_x \le [\;]_y \iff D_{(a,r)} \subseteq D_{(a',r')}$

(can extend this to Type IV).

So given x, y, can find $x \vee y$ in $(\mathbb{A}^1)^{an}$: $|\;|_{D(a,r)} \vee |\;|_{D(a',r')} = |\;|_{D(a,|a-a'|)}$ (if not related)

• **Metric outside Type I** If $x \le y$ $\quad d([\;]_x,[\;]_y) = \left|\log\left(\frac{\operatorname{diam}[\;]_y}{\operatorname{diam}[\;]_x}\right)\right|$

where $\operatorname{diam}[\;]_x = \lim r_i$ if $x \leftrightarrow \{D(a_i,r_i)\}$.

In general: $d([\;]_x,[\;]_y) = d([\;]_x,[\;]_{x\vee y}) + d([\;]_{x\vee y},[\;]_y)$

• Type I points are infinitely far away.

Eg: $K = \mathbb{C}_p$