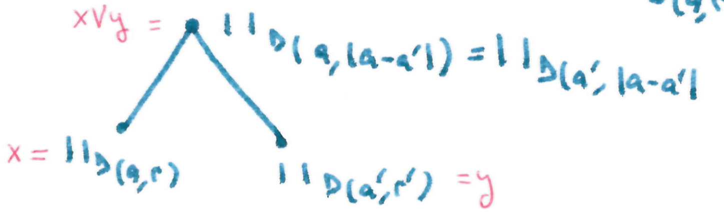
Lecture XXIX : Berkovich analytic spaces III

Recall: Constructed $(\mathbb{A}^1)^{an}$ by given explicit (semi) norms on $K[T]$

- From $D(a,r) = \{ z \in K \mid |a-z| \le r \} \subseteq K$ for $a \in K$, $r \ge 0$, we constructed $| \ |_{D(a,r)} : K[T] \longrightarrow \mathbb{R}_{\ge 0}$ (sup-norm)

$$f \longmapsto \sup_{z \in D(a,r)} |f(z)|$$

- Non-Archimedean norm on K induces partial order on $\{| \ |_{D(a,r)}\}_{a \in K, r \ge 0}$ & paths



- $[\ ]_x \le [\ ]_y \iff x = D(a,r) \subseteq D(a',r') = y$.
- Nested sequences of discs $\{D(a_i,r_i)\}_i$ give rise to $\| \ \| = | \ |_{\{D(a_i,r_i)\}} \in (\mathbb{A}^1)^{an}$ $\quad \|f\| = \lim_{i \to \infty} |f|_{D(a_i,r_i)}$

Berkovich classification Thm: Points in $(\mathbb{A}^1)^{an}$ $\underset{\|\ \|}{\longleftrightarrow}$ nested sequences of discs

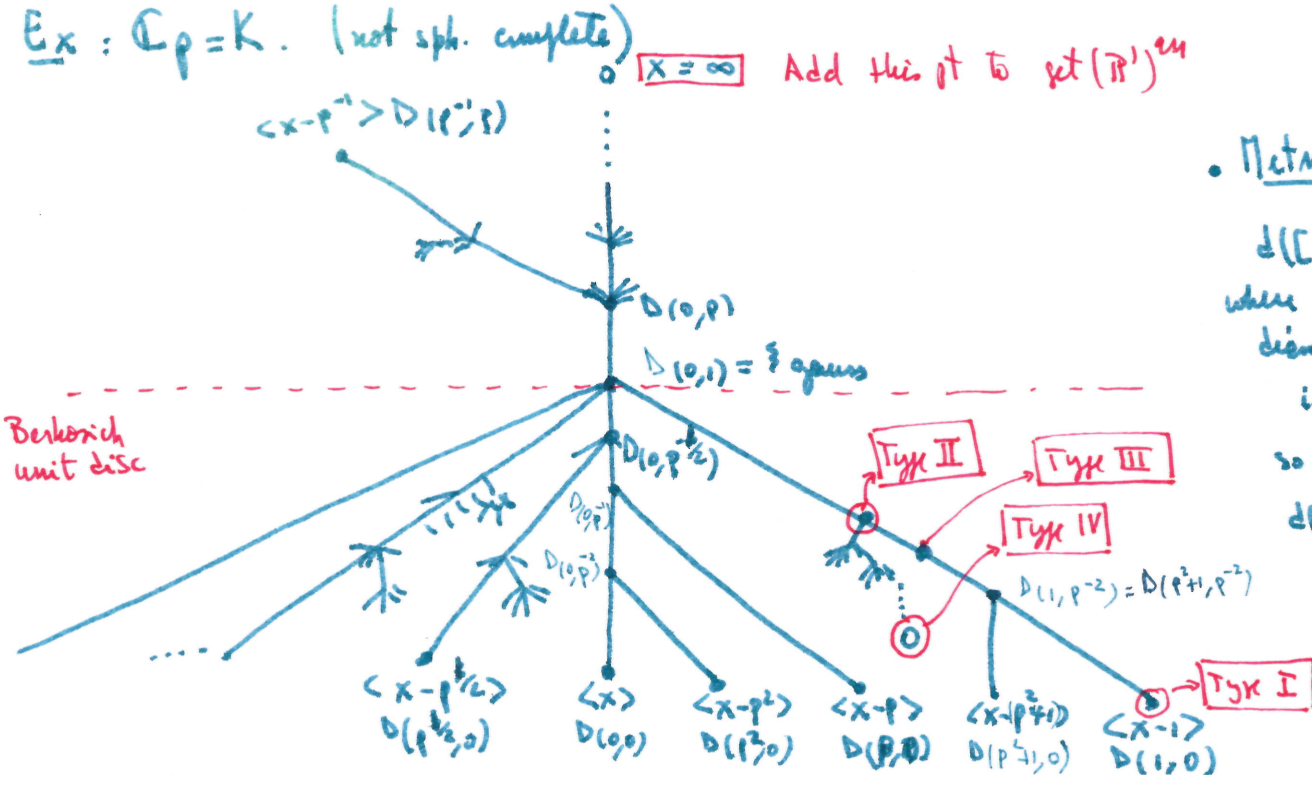
(1) Type I $\quad \lim_{n \to \infty} r_n = 0 \underset{K \text{ complete}}{\Longrightarrow} \bigcap_i D(a_i,r_i) = \{a\}$ & $\| \ \| = | \ |_{D(a,0)}$

(2) Type II $\quad \lim_{n \to \infty} r_n = r \in |K^\times|$ & $\exists\, a \in \bigcap_i D(a_i,r_i) \Rightarrow \| \ \| = | \ |_{D(a,r)}$

(3) Type III $\quad \lim r_n = r \notin |K^\times|$ & ________________

(4) Type IV : $\bigcap_i D(a_i,r_i) = \emptyset$ (so $\lim_{n \to \infty} r_n = r > 0$) $\rightsquigarrow$ only exists if K is not sph. complete.

Ex : $\mathbb{C}_p = K$. (not sph. complete)



- Metric outside Type I

$$d([\ ]_x,[\ ]_y) = \left| \log \frac{\operatorname{diam}(x)}{\operatorname{diam}(y)} \right|$$

where $\operatorname{diam}(x) = \lim r_i$ if $[\ ]_x \leftrightarrow \{D(a_i,r_i)\}$

so $d([\ ]_x,[\ ]_y) = d([\ ]_x,[\ ]_{x\vee y}) + d([\ ]_{x \vee y},[\ ]_y)$

Remarks: (1) $(\mathbb{A}^1)^{an}$ is an $\mathbb{R}$-tree but Topology induced by the metric outside Type I is FINER THAN the subspace Topology (coming from Berkovich Top)

Why? . Neighborhood of ξ_{gauss} = all but finitely many branches (which are replaced by open segments)

. with metric topology $\Rightarrow$ no Type I points in the nbhd of ξ_{gauss}.

[Can see this in trivial valn case (Lecture XXVII)]

(2) Branching at each pt x $\longleftrightarrow$ tangent vectors at x = equivalence classes of paths $[x,y]$ from x with $y \neq x$.

$[x,y_1] \sim [x,y_2]$ if they share a common initial segment

$[x,y]\sim[x,y']$
$(y \vee y' < x)$

(3) Type III pts = only 2 branches

(4) Type II pts = infinitely many branches $\longleftrightarrow$ $\mathbb{P}^1(\tilde{K})$ $\quad \tilde{K} = \mathcal{O}_v/\mathfrak{m}_v$.

(5) Proof of Classif Thm requires going to Banach algebras.

§2: Berkovich Unit Disc $= \{[\ ]_x : [\ ]_x \leq \xi_{gauss}\} = D(0,1)$

. Analytification of $K\langle T\rangle = \{\sum_{i=0}^{\infty} a_i T^i \in K[[x]] \mid \lim_{i\to\infty} |a_i| = 0\}$ = formal power series converging in $D(0,1)$.

Norm on $K\langle T\rangle$ = $\|f\| = \max_i |a_i|$ = Gauss norm $\qquad \exists C_x : [f]_x \leq C_x \|f\| \ \forall f$.

my $K^{an}_{\langle T\rangle} = \mathcal{M}(K\langle T\rangle) = \{[\ ]_x : K\langle T\rangle \to \mathbb{R}_{\geq 0}$ mult seminorms, bounded by $\|\ \|$ & $[0]_x = 0$, $[1]_x = 1$, $[f+g]_x \leq [f]_x + [g]_x\}$ with pointwise convergent Topology (via ev_f).

- Can always take $C_x = 1$ ($\|\ \|$ is multiplicative)
- $[a]_x = |a| \quad \forall a \in K$.
- Lemma: $[f+g]_x \leq \max\{[f]_x, [g]_x\}$ (non-Arch!) & = if $[f]_x \neq [g]_x$.
- For any $a \in D(0,1)$, $r \leq 1$: $|\ |_{D(a,r)} \in K\langle T\rangle^{an}$ because $|f|_{D(a,r)} = \sup_{z\in D(a,r)} |f(z)| = \sup_i |a_i| (\underbrace{\max\{r,|a|\}}_{\leq 1})^i \leq \|f\|$ by the Max Modulus Principle.

Topology with subbasis: $U(f,\alpha) = \{[\ ]_x \mid [f]_x < \alpha\}$, $V(f,\alpha) = \{[\ ]_x \mid [f]_x > \alpha\}$ $\quad \forall f \in K\langle T\rangle$, $\forall \alpha \in \mathbb{R}_{>0}$

- KEY: $[\ ]_x$ is determined by values on linear polynomials $T-a$ for a in $D(0,1)$ (Weierstrass Preparation Thm)

Weierstrass Prep Thm: Can factor any $f \in K\langle T\rangle$ uniquely as $f = c\prod_{j=1}^{m}(T-\alpha_j)\, u(t)$

where $\alpha_j \in D(0,1)$ $\forall j$ & $u(t)$ is a <u>unit</u> in $K\langle T\rangle$ with $u(t) = 1 + \sum_{i=1}^{\infty} a_i T^i$

(so $\|u(t)\| = 1$ & $[u]_x = 1$ $\forall [\,]_x$ by strong Δ-ineq) $\quad |a_i| < 1$ $\forall \geq i$, $\lim_{i\to\infty} |a_i| = 0$

<u>Proof of Berkovich classification on $D(0,1)$</u>: Pick $[\,]_x \in (\mathbb{A}^1)^{an}$. Furthermore, in $D(0,1)$.

$\mathcal{F} = \{D(a, [T-a]_x) : a \in D(0,1)\}$

<u>Note</u>: $[T-a]_x \leq \|T-a\| = \max\{1, |a|\} = 1$

- <u>Claim 1</u>: $\mathcal{F}$ is totally ordered by containment.

Pf/ If $[T-b]_x \geq [T-a]_x \Rightarrow |b-a| = [b-a]_x \leq \max\{[T-b]_x, [T-a]_x\} = [T-b]_x$ (*)

and $=$ if $[T-b]_x > [T-a]_x$.

So $a \in D(b, [T-b]_x)$ & $D(a, [T-a]_x) \subseteq D(b, [T-b]_x)$

- Set $\boxed{r_x := \inf_{a \in D(0,1)} [T-a]_x}$ & choose $a_i \in D(0,1)$ st $r_i = [T-a_i]_x \leq 1$ form a decreasing sequence converging to r

By (*) $D(a_1, r_1) \supseteq D(a_2, r_2) \supseteq \cdots$

- <u>Claim 2</u> $[T-a]_x = \lim_{n\to\infty} [T-a]_{D(a_n, r_n)}$ $\quad \forall a \in D(0,1)$

Pf/ • By definition of r: $[T-a]_x \geq r$ (& $[T-a]_{D(a_n, r_n)} \geq r_n$ $\forall n$)

• If $[T-a]_x = r \leq r_i$ $\forall i$ $\Rightarrow$ $\forall i \gg 1$ $r_i = [T-a_i]_x \geq |a_i - a|$

use (*) $b = a_i$

so $a \in D(a_i, r_i)$ $\forall i$ & $[T-a]_{D(a_i,r_i)} = \sup_{z \in D(a_i, r_i) = D(a, r_i)} |z-a| = r_i \longrightarrow r = [T-a]_x$

• If $[T-a]_x > r$, then for $i \gg 1$ $[T-a]_x > [T-a_i]_x$ &

again by (*) $[T-a]_x = |a-a_i|$. so $|a-a_i| > r_i$. for $i \gg 1$

Then $[T-a]_{D(a_i,r_i)} = \sup_{z \in D(a_i,r_i)} |z-a| = |a_i - a| = [T-a]_x \xrightarrow[i\to\infty]{} [T-a]_x$.

stable seq!

$\Rightarrow$ By 2 cases above: $[f]_x = \lim_{i\to\infty} [f]_{D(a_i,r_i)}$ $\forall f \in K\langle T\rangle$ & $D(a_i, r_i)$ is a nested sequence!

(A) If empty $\cap$ $\Rightarrow$ Type IV) $\forall i \to \infty$

(B) If the nested sequence has non-empty intersection, say $a \in \bigcap D(a_i, r_i) \subseteq D(0,1)$

then $r \leq [T-a]_x = \lim_{i\to\infty} [T-a]_{D(a_i,r_i)} \leq \lim_{i\to\infty} r_i = r \Rightarrow [T-a]_x = r$

Then $D(a,r) = D(a, [T-a]_x) \in \mathcal{F}$ & it's <u>minimal</u> (by def of r.)

If $r > 0$ $\Rightarrow$ We can take $a_i = a$ & $r_i = r$ $\Rightarrow$ $[f]_x = |f|_{D(a,r)}$ $\rightsquigarrow$ Type II or III

inf in def of r is a min

• If $r = 0$, $[\,]_x$ is type I.