Lecture XXX: Berkovich analytic spaces IV

Recall: $K\langle T\rangle = \{ f = \sum_{i=0}^{\infty} a_i T^i \in K[\![T]\!] \mid \lim_{i\to\infty} |a_i| = 0 \}$ = formal power series convergent in $D(0,1)$

with Gauss norm $\|f\| = \max_{i\in\mathbb{N}_0} |a_i|$. (multiplicative)

$D(0,1)$ = Berkovich unit disc $= \mathcal{M}(K\langle T\rangle) = \{$ mult seminorms $: [\ ]_x : K\langle T\rangle \to \mathbb{R}_{\geq 0}$ bounded by $\|\ \|$ & $[0]_x = 0$, $[1]_x = 1$, $[f+g]_x \leq [f]_x + [g]_x \}$

Subbasis (Topology): $U(f,\alpha) = \{ [\ ]_x \mid [f]_x < \alpha \}$
$V(f,\alpha) = \{ [\ ]_x \mid [f]_x > \alpha \}$ $\quad \forall f \in K\langle T\rangle$, $\alpha \in \mathbb{R}_{\geq 0}$.

Thm: Classification Thm of Berkovich holds for $D(0,1)$. (proof given last time)

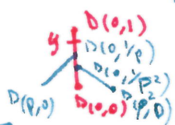
Use this idea to prove the result for $(\mathbb{A}^1)^{an}$:

- General Berkovich discs $D(0,R) = \mathcal{M}(K\langle R^{-1}T\rangle)$
  where $K\langle R^{-1}T\rangle = \{ \sum a_i T^i : \lim_{i\to\infty} R^i a_i = 0 \}$ ← convergent series in $D(0,R) \subset K$ is a Banach algebra
with Banach (multiplicative) norm $\|f\|_R = \max_i R^i |c_i|$.
- By construction: $(\mathbb{A}^1)^{an} = \bigcup_{R>0} D(0,R)$ via natural inclusions for $r<R$
  $K\langle R^{-1}T\rangle \hookrightarrow K\langle r^{-1}T\rangle$
- Theorem holds for all $D(0,R)$, hence holds for $(\mathbb{A}^1)^{an}$ ✓

Prop: $D(0,1) = \varprojlim \Gamma$ for $\Gamma = \bigcup_{i=1}^{n} [\,|\ |_{D(a_i,r_i)}, \xi_{gauss}]$ $\quad a_i \in D(0,1)$, $r_i \leq 1$.
(where $\xi_{gauss} = |\ |_{D(0,1)}$)

Why? Here $\Gamma \subsetneq \Gamma'$ gives a retraction map $r_{\Gamma',\Gamma} : \Gamma' \longrightarrow \Gamma \ni y$
$x \longmapsto$ first point where the path $[x,y]$ meets Γ

(The map is independent of y) Eg



Note: • $r_{\Gamma',\Gamma}(x) = x \iff x \in \Gamma$

• Maps are compatible $\Gamma \leq \Gamma' \leq \Gamma''$ $\quad r_{\Gamma'',\Gamma} = r_{\Gamma'',\Gamma'} \circ r_{\Gamma',\Gamma}$.

§1: What is X^{an} for arbitrary curves X/K?

• For simplicity, assume X is smooth & complete curve $/K$, $K = \overline{K}$.

• Gluing of affine schemes: X^{an} is locally homeomorphic to $(\mathbb{A}^1)^{an}$, with global topology captured by a finite PL object Γ = skeleton. (It's a metric graph)

- Skeletons Γ = dual graphs of central fibers of semistable regular models of $X/\mathcal{O}_v$
- $\exists$ retraction map $X^{an} \twoheadrightarrow \Sigma$ & it's a strong deformation retract.

Def: A curve C over a field $F = \overline{F}$ is semistable if it is reduced & its singular points are nodal.

Def: A ~~semistable~~ regular model of $X/\mathcal{O}_v$ is a finite type regular scheme $\mathfrak{X}/\mathcal{O}_v$ and a flat map

$$\begin{array}{c} \mathfrak{X} \\ \pi \downarrow \\ \operatorname{Spec}\mathcal{O}_v \end{array}$$

so that $\pi^{-1}_{(\eta)} = \mathfrak{X}_\eta = X$ for η generic pt of $\operatorname{Spec}\mathcal{O}_v$ $(\eta = (0))$

$\pi^{-1}_{(0)} = \mathfrak{X}_0$ is a semi-stable curve over $\widetilde{K} = \mathcal{O}_v/\mathfrak{m}_v$

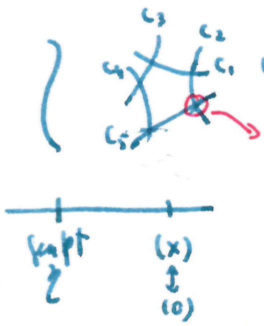
Example: E elliptic curve with bad reduction $(\Leftrightarrow \operatorname{val}(j_E) < 0)$

$\mathfrak{X}$

$\downarrow$

$\operatorname{Spec}\mathcal{O}_v$

t = uniformizer

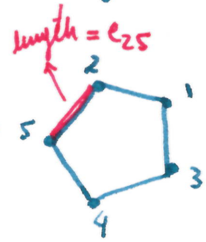


wheel of $\mathbb{P}^1$'s

local eqn $xy = t^{e_{25}}$

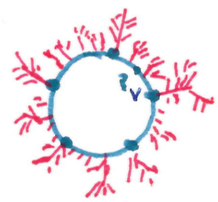
$e_{25} \geq 0$

$\rightsquigarrow$ dual graph Γ = S^1



$\text{length}(S^1)$ = sum of 5 edges $= -\operatorname{val}(j_E)$

Picture of E^{an}:



2 local pictures:

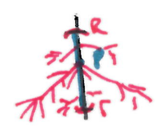
(1) Around a vertex v in S^1: $= D^{op}(f,\alpha) = \{[\ ]_x : [f]_x < \alpha\}$ open disc

v behaves like ζ_{gauss}

(2) Around a point in an edge $= \operatorname{Ann}(f, r, R)$
$= \{[\ ]_x : r < [f]_x < R\}$

Skeleton of $D(f,\alpha)$ = 1 pt = vertex v

Skeleton of $\operatorname{Ann}(f,r,R)$ = a segment of length $= \log R - \log r$



I skeleton

- **Method 1**: models for X are constructed using Semistable Reduction for curves (normalizations + base changes)
- Semistable decomposition [Baker-Payne-Rabinoff] $\exists$ finite set $V \subseteq X^{an} \setminus X(K)$ with

$$X^{an} \setminus V \simeq \bigsqcup_{\text{finite}} \operatorname{Ann}(f,r,R) \sqcup \bigsqcup_{\text{finite}} D^{op}_{(f_i,\alpha_i)}$$

D^{op}_i = open discs

Ann = open annuli.

and the induced metric on $X^{an} \setminus X(K)$ is independent of V

The Skeleton $\Sigma(X,V) = \bigsqcup_{\text{finite}}$ segments $\sqcup \bigsqcup_{\text{finite}} \{pts\}$ (pts are the ends of the segments) $\{pts\} = V$.

Note: Connected components of $X^{an} \smallsetminus \Sigma(X,V)$ = open Berkovich discs with a unique pt in its closure ($\simeq \{$gauss point in $\{[\ ]_x : [T]_x < 1\}$ )

Retraction: $X^{an} \twoheadrightarrow \Sigma(X,V)$ maps each component to the ! $[\ ]_z$ in its closure

- **Method 2**: Given a parameterization of X, we can build X^{an} & its skeleton from the zeroes & poles of the map (join them by paths!)

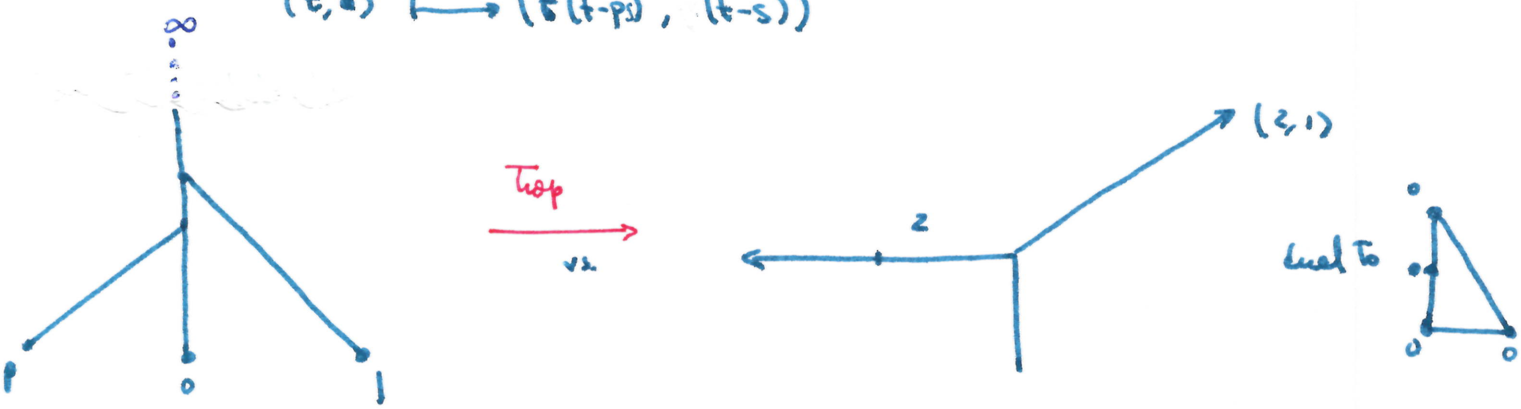
Example 2: $K = \mathbb{C}_p$ $\quad X \subseteq (K^*)^2$ defined by $x(t) = t(t-p) = f$

$y(t) = t - 1 = g$

$K[t] \xrightarrow{(f,g)} K[x,y]$ $\qquad (X = \mathbb{A}^1 \smallsetminus \{0,1,p\}) \rightsquigarrow X = V(y^2 + (2-p)y - x + p - 1)$

Extend to $\mathbb{P}^1 \longrightarrow \mathbb{P}^2$ $\qquad$ Zeros & poles = $0, 1, p, \infty$

$(t,s) \longmapsto (t(t-ps), (t-s))$



§2 Tropical shadows: Fix $X \xhookrightarrow[cl]{i} (K^*)^n = \operatorname{Spec} K[y_1^{\pm}, \dots, y_n^{\pm}]$ (or $X \hookrightarrow Y_\Sigma \supseteq (K^*)^n$)

$\exists\, (\mathrm{trop}_i) \; \| \ \| \longmapsto (\log\|y_1\|, \dots, \log\|y_n\|)$: $X^{an} \dashrightarrow \operatorname{Trop}(X,i)$ $\quad (*)$

$\| \ \|_p : f \mapsto |f(p)| \quad \longleftarrow \quad p$

$X(K) \hookrightarrow X^{an}$

$X(K) \xrightarrow{\varphi} \operatorname{Trop}(X,i) \subseteq \operatorname{Trop}(K^*)^n = \mathbb{R}^n$

$p \longmapsto (-\operatorname{val}(p_1), \dots, -\operatorname{val}(p_n))$

$= (\log|y_1(p)|, \dots, \log|y_n(p)|)$

Why?: Assume $K = \overline{K}$, val nontrivial (b/c $\operatorname{Trop}(X,i)$ doesn't change & $X_L^{an} \twoheadrightarrow X^{an}$ if $L|K$ extension)

Then $\overline{X(K)} = X^{an}$ & $\overline{\operatorname{im}\varphi} = \operatorname{Trop}(X,i)$ by Kapranov's Thm (Fund Thm of Trop Geom)

Key: $\mathrm{trop}: X^{an} \longrightarrow \operatorname{Trop}(X,i)$ is continuous & SURJECTIVE, even if val on K is trivial! [discussion above]

- The construction extends to $X \xhookrightarrow{i} Y_\Sigma$ Toric Varieties.