Lecture XXXI: Berkovich analytic spaces V

Last Time: $X \underset{cl}{\overset{i}{\hookrightarrow}} Y_\Sigma$, $\operatorname{Spec} K[y_1^{\pm},\ldots,y_n^{\pm}] = (K^\times)^n$

$$X^{an} \xrightarrow[\text{[CONT]}]{(\mathrm{Trop},i):\ \|\ \| \mapsto (\log\|y_1\|,\ldots,\log\|y_n\|)} \mathrm{Trop}(X,i) \subseteq \mathrm{Trop}(Y_\Sigma)$$

$X(K) \hookrightarrow X^{an}$, $X(K) \longrightarrow \mathrm{Trop}(X,i)$

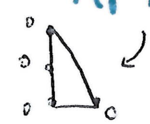
$z \longmapsto (\log|y_1(z)|,\ldots,\log|y_n(z)|) = (-\mathrm{val}(z_1),\ldots,-\mathrm{val}(z_n))$

Why? If K non-trivial valn, $\overline{K}=K$, $\overline{X(K)} = A^n$ so use Kapranov's Thm

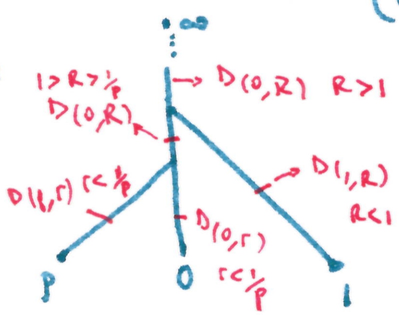
Example: $X \overset{i}{\hookrightarrow} (K^\times)^2$ defined by $\begin{cases} f(t) = t(t-p) \\ g(t) = t-1 \end{cases}$ $\quad X = V(y^2+(2-p)y-x+(p-1))$

$K=\mathbb{C}_p$, $X = \mathbb{A}^1 \smallsetminus \{0,1,p\}$

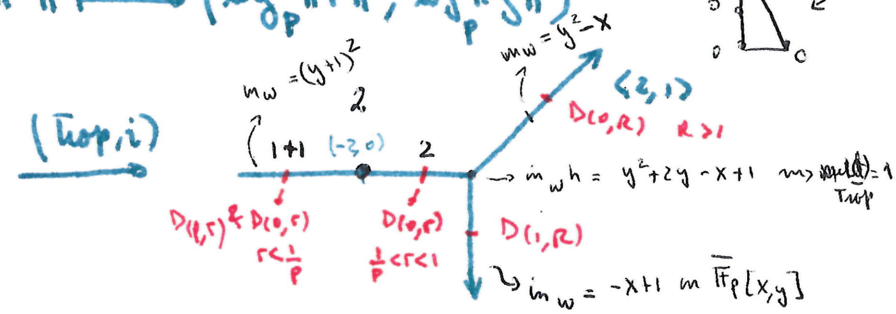
$$(\mathrm{Trop},i): \|\ \| \longmapsto (\log_p\|f\|, \log_p\|g\|)$$



Saw: Skeleton



$\xrightarrow{(\mathrm{Trop},i)}$



Explicitly:

- $R>1$: $|f|_{D(0,R)} = |t|_{D(0,R)}\,|t-p|_{D(0,R)} = R^2 \longmapsto 2\log_p R$
  $|g|_{D(0,R)} = |t-1|_{D(0,R)} = R \longmapsto \log_p R$, with $D(0,R) = D(1,R)$
  $\Big\}$ pt $= \underbrace{\log_p R}_{\geq 0}(2,1)$

  $|x-\alpha|_{D(a,R)} = \begin{cases} |a-\alpha|_p & \text{if } \alpha \notin D(a,R) \\ R & \text{o/w} \end{cases}$

- $\frac{1}{p}<R<1$: $|f|_{D(0,R)} = R^2 \longrightarrow 2\log_p R$
  $|g|_{D(0,R)} = |1-0|_p = 1 \longmapsto 0$ $\quad (\|t\| = R)$
  $\Big\}$ pt $= 2\log_p R\,(1,0) = 2\underbrace{(-\log_p R)}_{\in(0,1)}\langle -1,0\rangle$.

- $R<\frac{1}{p}$: $|f|_{D(0,R)} = R\,|p-0| = \frac{R}{p} \longmapsto \log_p R - 1$
  $|g|_{D(0,R)} = 1 \longmapsto 0$
  $\Big\}$ pt $= (\log_p R - 1, 0) = \underbrace{(-\log_p R)}_{>1}(-1,0) + (-1,0)$

- $R<\frac{1}{p}$: $|f|_{D(p,R)} = R/p$, $|g|_{D(p,R)} = 1$, so same images as ↗

- $R<1$: $|f|_{D(1,R)} = 1^2 \longmapsto 0$
  $|g|_{D(1,R)} = R \longmapsto \log_p R$
  $\Big\}$ pt $= (0,\log_p R) = \underbrace{-\log_p R}_{>0}(0,-1)$

$\mathrm{Trop}(\overline{\mathrm{Star}_{X^{an}}\, v})$ is a balanced fan. (tangent directions)

FACT: The map $(\mathrm{Trop},i): X^{an} \twoheadrightarrow \mathrm{Trop}(X,i)$ for X curve is piecewise linear & harmonic with integer slopes & $\mathrm{mult}_{\mathrm{Trop}}(e) = \sum\limits_{\substack{f \text{ edge in } X^{an} \\ \mathrm{Trop}(f)=e}}$ stretching factor for f (X^{an} = locally an $\mathbb{R}$-tree)

stretching factors = slopes; e = edge

↪ POINCARÉ-LELONG FORMULA

- If edges get contracted, then stretching factor $= 0$.
- By choosing a skeleton $\Sigma(X,V)$ containing all V pts in $X(K)$ at the boundary of $X \hookrightarrow Y_\Sigma$ where $\Sigma = \mathrm{Trop}_{\text{trivial}} X$ (as in the example), we get a commuting diagram

$$\begin{tikzcd} X^{an} \arrow[d, two heads, "\text{retr.}"'] \arrow[dr, two heads, "\text{trop}"] & \\ \Sigma(X,V) \arrow[r, two heads, "\text{PL map}"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{Trop}(X,i) \end{tikzcd}$$

$\Sigma(X,V) = $ graph

Everything in $X^{an} \setminus \Sigma(X,V)$ gets contracted.

For general X:

Q: What happens to the diagram
$$\begin{tikzcd} X^{an} \arrow[dr, two heads, "\mathrm{trop}_i"] & \\ X(K) \arrow[u] \arrow[r] & \mathrm{Trop}(X,i) \end{tikzcd}$$
under equivariant $\mathcal{U}$-embeddings of $X \overset{i}{\hookrightarrow} Y_\Sigma$?

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "i'"'] & Y_\Sigma \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] & \supset (K^*)^n \arrow[d] \\ & Y_{\Sigma'} & \supset (K^*)^m \end{tikzcd}$$

monomial map ass. to $A \in \mathbb{Z}^{m \times n}$ $\rightsquigarrow$ $\boxed{\mathrm{Trop}(X,i') = A\,\mathrm{Trop}(X,i)}$

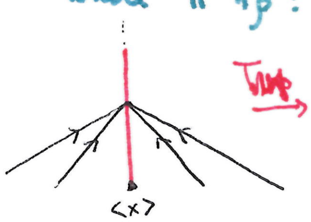
$\rightsquigarrow$ get an inverse system of embeddings!

Thm [Payne '09] $X^{an} \underset{\text{homeo}}{\simeq} \varprojlim_{X \overset{i}{\underset{\mathcal{U}}{\hookrightarrow}} Y_\Sigma} \mathrm{Trop}(X,i)$

Example: $\mathbb{A}^1$ with trivial valuation: $\mathbb{A}^1 \overset{i}{\hookrightarrow} \mathbb{A}^1$ $\mathrm{Trop}(X,i) = \overline{\mathbb{R}} \hookrightarrow X^{an}$

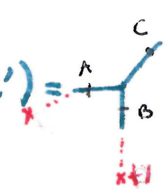
$\rho \longmapsto \|\ \|_\rho$

where $\|\ \|_\rho : \sum a_i t^i \longmapsto \max_{a_i \neq 0} \{\underbrace{|a_i|}_{=1} \exp(\rho)^i\}$ [Skeleton norm]

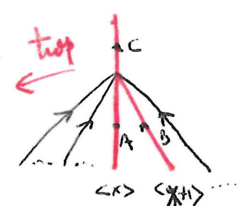


$\xrightarrow{\text{Trop}} \overline{\mathbb{R}}$

Now: $\mathbb{A}^1 \overset{i'}{\hookrightarrow} \mathbb{A}^2$ $\quad y = x+1$ $\quad \mathrm{Trop}(X,i') =$
$x \longmapsto (x, x+1)$



$\xleftarrow{\text{trop}}$



$\Rightarrow$ We add one more dimension per branch and in the limit we get $(\mathbb{A}^1)^{an}$.

Q2: Can we see $\mathrm{Trop}(X,i)$ as a closed subset of X^{an} for some i via a continuous section to $X^{an} \underset{(\mathrm{trop}(i))}{\overset{\sigma}{\rightleftarrows}} \mathrm{Trop}(X,i)$? If so, call tropicalization faithful.

Why do we expect this?

Thm (Hrushovski-Loeser '10) X quasi-projective. Then X is locally contractible & homotopy equivalent to a finite simplicial complex of dim $=$ dim X (Skeleton of X^{an})
(complex $=$ dual complex associated to a semistable regular model of X)

AIM: Use Trop geometry to build this complex & understand how it maps to $\mathrm{Trop}(X,i)$

Q3: Can we detect faithfulness solely from $\mathrm{Trop}(X,i)$ or local information on initial degenerations of X, but without knowing X^{an}?

Q4: If we detect NONfaithfulness, can we repair the embedding? Effectively?
[C-Markwig]

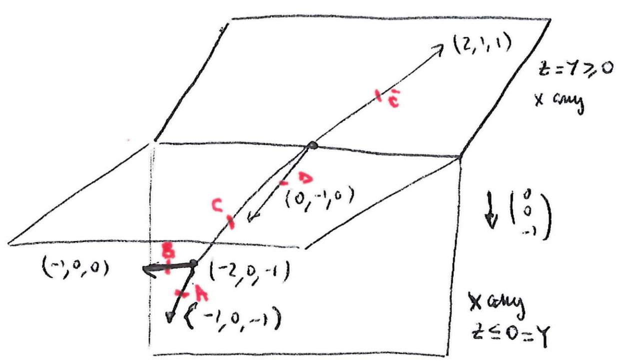
Thm [Baker-Payne-Rabinoff '11]: For any finite embedded graph $\Gamma \subset X^{an} \setminus X(K)$, $\exists$ an embedding $i : X \hookrightarrow Y_\Sigma$ st (1) trop maps Γ piecewise linearly & ISOMETRICALLY onto its image
(2) Each edge in $\mathrm{trop}^{-1}(\mathrm{trop}\,\Gamma)$ that is disjoint from Γ is contracted to a point.
The system of all such embeddings is stable & cofinal.

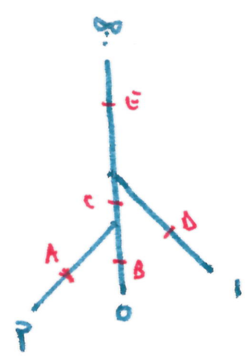
Corollary: If $\Gamma' \subset \mathrm{Trop}(X,i)$ & $m_{\mathrm{Trop}}(w)=1$ $\forall\, w \in \Gamma'$ (including edges & vertices) then, there exists a! $\Gamma \subset X^{an} \setminus X(K)$ subgraph mapping isometrically onto Γ'.

Example 1 (cont.) Know it's not faithful, how to repair it? Look at
$m_{\mathrm{Trop}}(w) > 1$ locus $\rightsquigarrow$ $\mathrm{in}_w = (y+1)^2$

Re-embed in $(K^*)^3$ via $I = \langle h, z-(y+1) \rangle$ $K[t] \longrightarrow K[x,y,z]$ $(t, y, y+1)$

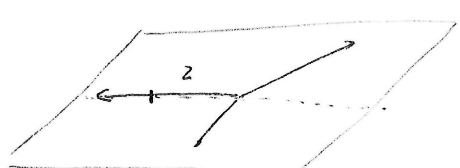
$\mathrm{Trop}(X)$ lives in $\mathrm{Trop}(z-(y+1))$





Why? $y+1$ becomes a unit $(=z)$ & we reduce the tropical multiplicity
&
General method? [C.-Markwig]

$\downarrow \pi_{xy}$



TROPICAL MODIFICATION of $\mathbb{R}^2$ along $\mathrm{Trop}(y+1) = \max\{Y, 0\}$

- Elliptic curves? Other curves? High dim'l varieties?
  - ↳ [BPR]
  - ↳ · no metrics
    · skeletons are more difficult to build [Gubler-Werner-Rabinoff]