# Lecture XXXII: Berkovich analytic spaces VI

**Recall:**

$$\begin{array}{ccc} X^{an} & \xrightarrow{(\mathrm{trop},i)} & \\ \uparrow & \circlearrowleft & \\ X(K) & \longrightarrow & \mathrm{Trop}(X,i) \subseteq \mathrm{Trop}\, Y_\Sigma \\ p & \longmapsto & (-\log|y_1(p)|, \dots, \log|y_n(p)|) \end{array}$$

$\| \; \| \mapsto (\log\|y_1\|, \dots, \log\|y_n\|)$

$\varprojlim_{X \overset{i}{\hookrightarrow} Y_\Sigma} \mathrm{Trop}(Y,i) \underset{\text{homeo}}{\simeq} X^{an}$

$X \underset{cl}{\overset{i}{\hookrightarrow}} Y_\Sigma \supset (K^\times)^n = \mathrm{Spec}\, K[y_1^\pm, \dots, y_n^\pm]$

. For $X$ curve, we can find a vertex set $V \subset X^{an} \setminus X(K)$ s.t. $\Sigma(X,V) =$ finite graph $\cup$ finite ends

where

$$\begin{array}{ccc} X^{an} & \xrightarrow{(\mathrm{Trop},i)} & \text{PL with } \mathbb{Z}\text{-slopes} \\ \text{retr} \downarrow & & \\ \Sigma(X,V) & \longrightarrow & \mathrm{Trop}(X,i) \end{array}$$

. For general $X$: If $\exists$ cont section to trop $\sigma: \mathrm{Trop}(X,i) \longrightarrow X^{an}$ we say $i$ induces a faithful tropicaliz.

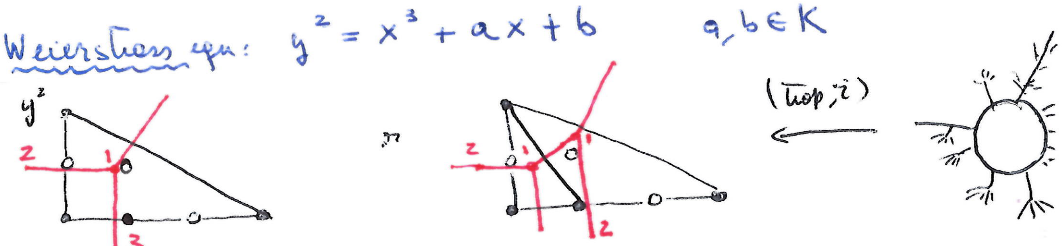
. BPR Thm: If $\Gamma' \subset \mathrm{Trop}(X,i)$ & $m_{\mathrm{Trop}}(\omega) = 1$ $\forall\, \omega \in \Gamma'$ (edges & vertices), then there exists a ! $\Gamma \subset X^{an} \setminus X(K)$ subgraph mapping isometrically onto $\Gamma'$.

## §1 Elliptic curves/$K$:

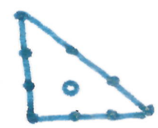
$E$ smooth elliptic curve with bad reduction, ie $\mathrm{val}(j_E) < 0$, $E \subset (K^\times)^2$ via a cubic eqn $g(x,y)$. $\mathrm{char}(K) \neq 2,3$

**Recall:** Minimal skeleton of $E^{an}$ = $S^1$ with length $= -\mathrm{val}(j_E)$.

∘ Weierstrass eqn: $y^2 = x^3 + ax + b \quad a, b \in K$




eg $\mathrm{val}(a) = \mathrm{val}(b) = 0$ eg $\mathrm{val}(a) \geq 0$, $\mathrm{val}(b) < 0$

. For $g$ cubic (irreducible), then $\mathrm{Trop}(g)$ is dual to a subdivision of 
& we have a chance of having a genus 1 graph (surrounding (1,1))
Also: edges in the loop have mult $= 1$ (dual to an internal edge in the subdivision containing the vertex (1,1)).

**Thm** [Katz-Markwig$^2$ '07] Trop $E$ has no loops (as a graph) or the loop has lattice length $\leq -\mathrm{val}(j_E)$
Equality holds if the vertices on the loop have valency $= 3$.

Why? . Stretching factors on edges in $S^1$ mapping to the loop are 0 or 1.

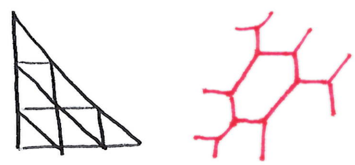
- $j_E = \dfrac{A}{\Delta}$ ↝ discr. Lemma: $-\mathrm{val}\, j_E$ goes up $\Leftrightarrow$ $\mathrm{val}(\Delta)$ goes up.
- valency $= 3$ + adjacent to a $\mathrm{mult}_{\mathrm{Trop}} = 1$ edge $\Rightarrow$ $m_{\mathrm{Trop}}(v) = 1$.

Here: $\mathrm{mult}_{\mathrm{Trop}}(w) = \#\{$irred comp of $\mathrm{in}_w(I) \subseteq \tilde{K}[y_1^\pm \dots y_n^\pm]\}$ (counted w/ multiplicity)

Q: How to repair when the inequality is strict?

<u>Thm</u>: [Chen-Sturmfels '11] $E$ can be reembedded in $(K^*)^3$ with $\mathrm{Trop}\, E$ in honeycomb form

↳ non-effectively!



<u>Thm</u> [C-Markwig '16] Whenever $\mathrm{Trop}\, E \subseteq \mathbb{R}^2$ either

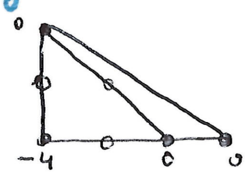
(1) has no loop & has a bounded edge $e$ of $m_{\mathrm{Trop}}(e) > 1$

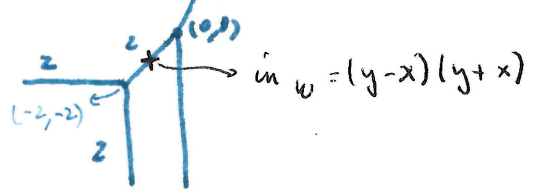
or (2) has a loop with a vertex $v$ with $m_{\mathrm{Trop}}(v) = 2$ dual to a trapezoid

($\Leftrightarrow$ non-trivalent)

Then, we can $\widehat{\text{linearly}}$-re-embed in dim $\leq 4$ & see a loop of length $= -\mathrm{val}(j_E)$

Eg: $K = \widehat{\mathbb{C}\{\{t\}\}}$ $\quad E = V(y^2 - x^3 - x^2 - t^4)$ $\quad \mathrm{val}(j_E) = -4$






re-embed via $I = \langle g, z-(y-x)\rangle$


length $= 2 \cdot 2 = \boxed{4}$

All multiplicities on the loop $= 1$.

## §2 Higher dimensions:

<u>Example</u> Skeleton norms on $(\mathbb{A}^n)^{an}$ induced by $(\overline{\mathbb{R}})^n \hookrightarrow (\mathbb{A}^n)^{an}$

given $\rho \in (\overline{\mathbb{R}})^n \rightsquigarrow \delta(\rho)$ multiplicative (semi)-norm on $K[x_1, \ldots, x_n]$

$$\delta_{(\rho)} : K[x_1, \ldots, x_n] \longrightarrow \mathbb{R}_{\geq 0}$$

$$f = \sum_{\substack{\alpha \\ \text{finite}}} c_\alpha x^\alpha \longmapsto \max_\alpha \{ \underset{\exp(-\mathrm{val}(c_\alpha))}{|c_\alpha|} \exp(\underset{\langle \alpha, \rho\rangle}{\textstyle\sum \alpha_i \rho_i}) = \underbrace{\exp(\underbrace{\mathrm{trop}(f)_{(\rho)}}_{\text{in } \overline{\mathbb{R}}})}_{\text{in } \mathbb{R}_{\geq 0}}$$

<u>Key properties</u>:

(1) Each $f$ has a unique polynomial representative $\Rightarrow \delta(\rho)$ is well-defined, it's multiplicative & with kernel $= \{0\}$

(2) $\delta(\rho)(x_i) = \exp(\rho_i)$ $\forall i$ so $\mathrm{trop}(\delta(\rho)) = \rho$.

(3) $\delta(\rho)$ is the maximal among all $\| \|$ with $\mathrm{trop}(\| \|) = \rho$.

($\rightsquigarrow$ The fiber $\mathrm{trop}^{-1}(\rho)$ has a distinguished pt, called it's ! Shilov boundary pt, inducing $\sigma: \mathrm{Trop}(\mathbb{A}^n) = \overline{\mathbb{R}}^n \hookrightarrow (\mathbb{A}^n)^{an}$ continuous section to $(\mathrm{trop}, i)$ ($i = \mathrm{id}_{\mathbb{A}^n}$))

**Thm** [C-Häbich-Werner '13, Draisma-Postinghel '14] The Plücker embedding $\varphi : Gr(2,n) \hookrightarrow \mathbb{P}^{\binom{n}{2}-1}$ induces a faithful tropicalization.

Furthermore: ① all trop multiplicities on Trop $Gr(2,n)$ equal 1.

Equivalently, if $Gr_{\mathcal{J}}(2,n) = \{ p \in Gr(2,n) : p_I = 0 \iff I \in \mathcal{J}^c \}$ (a matroid strata)

this $m_{Trop} = 1$ condition says $\underset{\pi_{\mathcal{J}}(w)}{\mathrm{in}} \underbrace{Gr_{\mathcal{J}}(2,n)}_{\cap \ (K^*)^{\mathcal{J}^c}/K^*} \subseteq \tilde{K}[p_B^{\pm} \mid B \in \mathcal{J}]$ is prime

Here: $\pi_{\mathcal{J}} : T\mathbb{P}^{\binom{n}{2}-1} \dashrightarrow T\mathbb{P}^{|\mathcal{J}|-1}$

② $\forall\, w \in \mathrm{Trop}\, Gr(2,n)$ : $\mathrm{trop}^{-1}(w) \subseteq Gr(2,n)^{an}$ has a ! distinguished pt $p$ satisfying $\|f\| \leq p(f)$ $\forall\, \|\ \| \in \mathrm{trop}^{-1}(w)$ & $\forall f \in K[p_B]/I_{2,n}$ "Shilov Boundary pt"

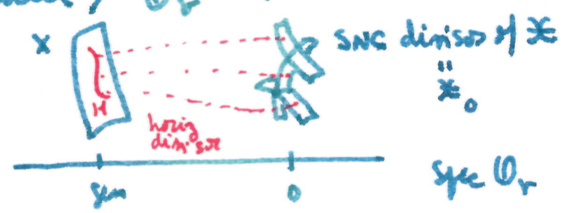
$\Rightarrow$ The cont section to trop sends $w$ to $p$.

**Thm** [Gubler-Rabinoff-Werner '14] If $X \overset{i}{\hookrightarrow} (K^*)^n$ $X$ irred & ALL trop mult $=1$, then each $\mathrm{trop}^{-1}(w)$ has a ! Shilov boundary pt & this defines a cont section to trop.

HARD: Extending this result to toric varieties is very delicate & often fails.

[GRW '15] Sufficient conditions involving behavior at infinity of cells in $\mathrm{Trop}(X \cap (K^*)^n)$

**Open problem**: Is the image of $\sigma$ a skeleton of $X^{an}$? ($\iff$ coming from a model $\mathfrak{X}/\mathcal{O}_v$ + a Cartier divisor $H$ on $X$. Skeleton = dual complex to $\mathfrak{X}_0$.




$X^{an} \setminus H^{an} \twoheadrightarrow S(\mathfrak{X}, H)$ (extended skeleton) canonical retraction map.

$S(\mathfrak{X}, H) = \bigcup_{\Delta_T \subseteq \Delta(\mathfrak{X}_0)} \Delta_T \times \mathbb{R}^p_{\geq 0}$

$T$ a strata in $\Delta(\mathfrak{X}_0)$ $\rightsquigarrow \{H_{i_1}, \ldots, H_{i_p}\}$ meeting components of $\mathfrak{X}_s$ indexed by $T$

**Proof idea** [CHW]:

Construct an open affine cover $Gr(2,n) = \bigcup_{i<j} U_{ij}$ $\quad U_{ij} = \varphi^{-1}(\{p_{ij} \neq 0\})$

- $U_{ij} \simeq \mathrm{Spec}\, K\left[\frac{p_{ik}}{p_{ij}}, \frac{p_{jk}}{p_{ij}} : k \neq i,j\right] \simeq \mathbb{A}^{2(n-2)}$

Use skeleton norms on $\mathbb{A}^{2(n-2)}$. Need: $\boxed{\sigma(w)\left(\frac{p_{kl}}{p_{ij}}\right) = \exp(w_{kl} - w_{ij}) \ \forall\, kl \neq ij}$

to have a (cont) section to trop on $\mathrm{Trop}\, U_{ij} \subseteq T\mathbb{P}^{\binom{n}{2}-1}$ (here $w_{ij} \neq -\infty$)

$\binom{n}{2}-1$ coords! $\frac{p_{kl}}{p_{ij}}$ on $U_{ij}^{an}$

$2(n-2)$ coords $\frac{p_{ik}}{p_{ij}}, \frac{p_{jk}}{p_{ij}}$

$$\begin{tikzcd}
Gr(2,n)^{an} \arrow[r, "Trop"] & Trop\, Gr(2,n) \arrow[l, dashed, bend left, "\exists ?\, \sigma \text{ cont}"] & \\
U_{ij}^{an} \arrow[u, phantom, "\cup"] \arrow[r, "Trop"] \arrow[dr, "Trop"] & Trop\, U_{ij} \arrow[u, phantom, "\cup"] \arrow[d, "\pi_{\{ik, jk \,:\, k \neq ij\}}"] & = \{ w \in Trop\, Gr(2,n) : w_{ij} \neq -\infty \} \\
& \overline{\mathbb{R}}^{2(n-2)} \arrow[ul, dashed, bend left] &
\end{tikzcd}$$

$\exists\ \sigma = S_{(p)}$ skeleton norm $p = (w_{kl} - w_{ij})$ $kl \in \{i-, j-\}$

- BUT from 3-Term Pl reln on $i\,j\,k\,l$

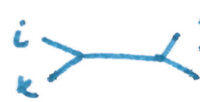
$$\frac{p_{kl}}{p_{ij}} = \frac{p_{ik}}{p_{ij}} \frac{p_{jl}}{p_{ij}} - \frac{p_{il}}{p_{ij}} \frac{p_{jk}}{p_{ij}}$$

Apply $\sigma(p)$:

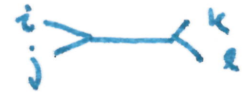
$$\sigma(p)\left(\frac{p_{kl}}{p_{ij}}\right) \underset{\text{def}}{=} \max\{ w_{ik} - w_{ij} + w_{jl} - w_{ij},\ w_{il} - w_{ij} + w_{jk} - w_{ij} \}$$

$$\overset{?}{/\!/}$$

$$w_{kl} - w_{il} \qquad = \max\{ w_{ik} + w_{jl},\ w_{il} + w_{jk} \} - 2 w_{ij}$$

$\Longleftrightarrow$ (1) $w \in \overline{\mathcal{C}_T}$ where $T$ contains 

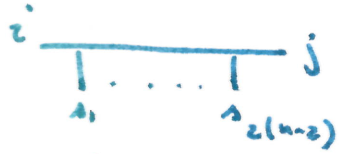
or 


but NEVER 


or

(2) One of $w_{ik}, w_{jk}, w_{il}$ or $w_{jl} = -\infty$.

- The conditions should hold $\forall\, k\, l$: On the torus $\mathbb{R}^{\binom{n}{2}} / \mathbb{R}\cdot 1$ this happens

$\Longleftrightarrow$ $T =$ 

caterpillar tree with backbone $i\,j$

- If $T$ is not a tree like this, then we must trade $\frac{p_{ik}}{p_{ij}}, \frac{p_{jl}}{p_{ij}}, \frac{p_{il}}{p_{ij}}$ or $\frac{p_{jk}}{p_{ij}}$ by $\frac{p_{kl}}{p_{ij}}$ (Eg if $w_{ik} \neq -\infty \rightsquigarrow \frac{p_{jl}}{p_{ij}} = \left(\frac{p_{ik}}{p_{ij}}\right)^{-1} \left( \frac{p_{kl}}{p_{ij}} + \frac{p_{il}}{p_{ij}} \frac{p_{jk}}{p_{ij}} \right)$

& we get $\sigma(w)\left(\left(\frac{p_{ik}}{p_{ij}}\right)^{-1}\right) = w_{ij} - w_{ik}$.

- Precise combinatorial rules (coming from trees & matroid strata $w \in Trop\, Gr_y(2,n)$) tell us how to pick our coordinates on $U_{ij}$.