Lecture XXXIV: Moduli spaces, curves & their tropical analogs

Recall: $M_{0,n}$ = iso classes of smooth genus 0 curves with n distinct marked pts $n \geq 3$

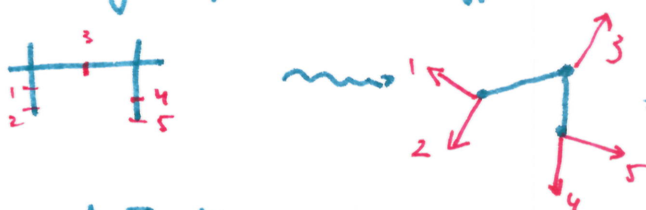
We compactify it by $\overline{M}_{0,n}$ = iso classes of stable rat'l curves $(C, p_1, \ldots, p_n)$

- C connected curve of arithmetic genus 0
- only nodal singularities
- $p_1, \ldots, p_n$ are distinct pts in $C \smallsetminus C^{sing}$
- each component of C has ≥ 3 special points (markings or nodes)

§1 Boundary of $\overline{M}_{0,n}$ = nodal curves

$$\boxed{\dim \overline{M}_{g,n} = 3g-3+n}$$

Stratified by combinatorial type = dual graphs of nodal curves with markings.

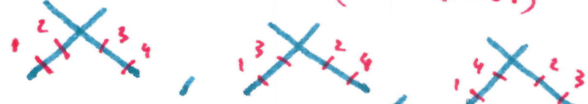
Eg: $C =$  $= \Gamma$

vertices = $\mathbb{P}^1$'s
edges = nodes where $\mathbb{P}^1$'s meet
half-rays = markings

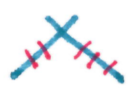
- Codimension of strata = # edges of Γ = # nodes of C
- Poset structure on strata ($S_1 < S_2 \iff S_1$ is in $\overline{S_2}$)
  $S_1 < S_2 \iff \Gamma_2$ is obtained from Γ_1 by contracting some edges. REVERSE THAN TREE SPACE!
- Strata are naturally isomorphic to products of $M_{0,n}$'s
- $\overline{\text{Strata}}$ ———————————— $\overline{M}_{0,n}$'s

} "inductive construction"

- We organize this information on the boundary complex (TREE SPACE!)
  { · k-cells = codim k strata
  · cells = glued via poset structure

Examples $\partial\overline{M}_{0,4} = \{$  $\}$

3 pts = $0, 1, \infty$ in $\mathbb{P}^1$! $M_{0,4} = \mathbb{P}^1 \smallsetminus \{0,1,\infty\}$.

$\partial\overline{M}_{0,5} = \{$  $\}$ ⇝ 10 copies of $M_{0,3} \times M_{0,4}$

codim 1 

$\cup \{$  $\}$ ⇝ 15 copies $\simeq M_{0,3}^3$ (in the closure of the 10 copies of $M_{0,3} \times M_{0,4}$)

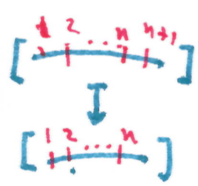
↳ Boundary complex = Petersen graph!

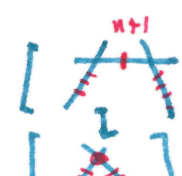
codim 2 

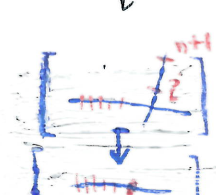
§2 Natural morphisms between $\overline{M}_{0,n}$'s:

① **Forgetful morphisms**

$$\begin{array}{ccc} \overline{M}_{0,n+1} \ni & [C, p_1, \ldots, p_{n+1}] \\ \downarrow f_{n+1} & \downarrow \\ \overline{M}_{0,n} \ni & [C, p_1, \ldots, p_n] \end{array}$$







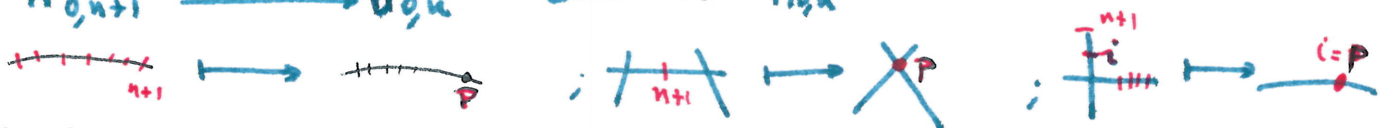
Issues on boundary! Forget the pt & contract components if necessary:

(1) contract 1 component if $n+1$ is the only pt in that $\mathbb{P}^1$

(2) If another single marking i exists, then contract the $\mathbb{P}^1$ to the node & assign the marking i to the resulting contraction (old node).

(2) Contractions & Stabilizations:

$$c: \overline{M}_{0,n+1} \longrightarrow \overline{U}_{0,n} \qquad \text{contraction} \qquad \begin{array}{c} U_{0,n} = M_{0,n} \times \mathbb{P}^1 \\ \downarrow \pi \\ M_{0,n} \end{array}$$

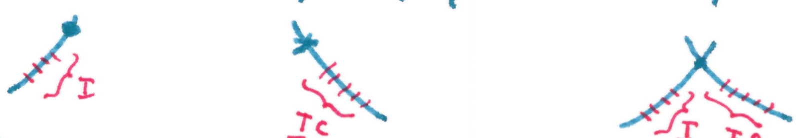


Stabilization = reverse the arrows!

$$\begin{tikzcd} \overline{M}_{0,n+1} \arrow[r, shift left, "c"] \arrow[dr, "f_{n+1}"'] & \overline{U}_{0,n} \arrow[l, shift left, "s"] \arrow[d, "\pi"] \\ & \overline{M}_{0,n} \end{tikzcd}$$

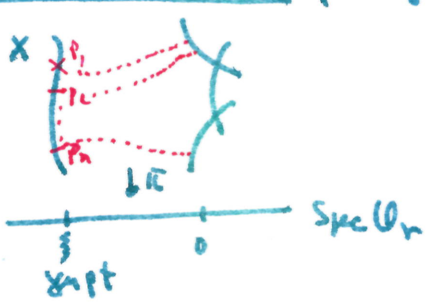
(3) Glueing morphisms: $I \subset \{1, \ldots, n\}$

$$gl_I : \overline{M}_{0, I \cup \{*\}} \times \overline{M}_{0, I^c \times \{*\}} \longrightarrow \overline{M}_{0,n}. \qquad \text{iso onto its image}$$



The image of gl_I is a boundary divisor D_I in $\overline{M}_{0,n}$ $\Rightarrow$ it is irreducible!

§3 Relation to models (strongly semistable)



$\mathfrak{X}$ = regular surface, proper, flat / $\mathcal{O}_K$

$\mathfrak{X}_K \simeq X$, $\mathfrak{X}_0$ is a reduced curve / $\tilde{K}$ with at worst nodes

Str ss components of $\mathfrak{X}_0$ have no self intersection

↳ no self loop in dual graph.

Our stable curves = special fibers

Markings? Horizontal divisors:

- $p_1, \ldots, p_n$ & see where they go to in the special fiber
- Condition = specialization to distinct smooth points in $\mathfrak{X}_0$ & ~~each~~ component of $\mathfrak{X}_0$ has ≥ 3 special pts

dual graphs = extended Berkovich skeleta (w.r.t. punctures $p_1, \ldots, p_n$)

§4 Other spaces?

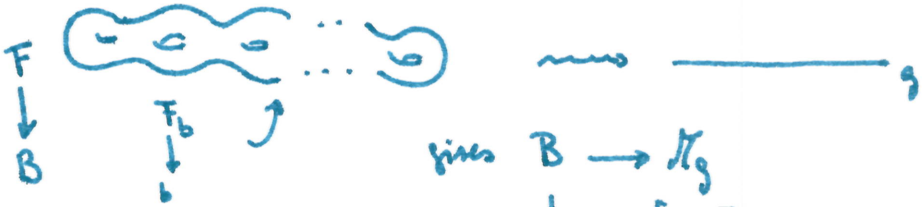
(I) $\overline{M}_{g,n} = \{ (C, p_1, \ldots, p_n)$ C connected, complete curve; C has genus g with at worst simple nodes, $p_1, \ldots, p_n \in C \smallsetminus C^{\text{sing}}$ markings; $(C, p_1, \ldots, p_n)$ has a finite # of automorphisms $\}$

$[2g-2+n>0]$

- Stratified by ~~combinatorial~~ type of dual graphs, but now vertices have weights = genus of the component
- Also: poset structure. ⇝ If contract an edge to a point: add weights of 2 vertices = weight pt; a loop: add 1 to the weight

(II) No markings:

pts on $\mathcal{M}_g$ = iso classes of smooth curves of genus g



gives $B \longrightarrow \mathcal{M}_g$
$b \longmapsto [F_b]$

$\overline{\mathcal{M}}_g$ = DM-compactification. pts are iso classes of stable curves with at worst simple nodal singularities

More precisely: $\overline{\mathcal{M}}_g \ni [C] \leftrightarrow$ C stable curve of genus g
C has at worst simple nodal singularities
C has a finite number of automorphisms.

In particular: • genus 0 components must meet the rest of the curve in ≥ 3 pts.
• genus 1 curves (elliptic tails) are attached to at least 1 pt

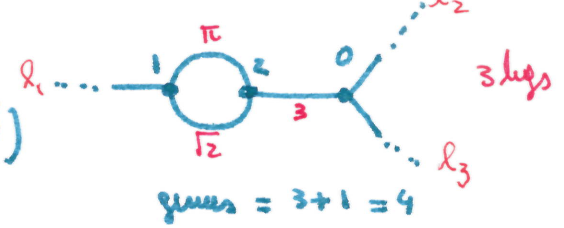
Stratify: $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g = \bigcup \{[C] \in \overline{\mathcal{M}}_g \mid C \text{ has} \geq 1 \text{ node}\} = \Delta_0 \cup \Delta_1 \cup \dots \cup \Delta_{\lfloor g/2 \rfloor}$

$\Delta_0 = \overline{\{$  $\}}$, $\Delta_i = \overline{\{$  $\}}$.

§5 Tropical analogs:

Def: A tropical curve Γ consists of:

- a finite graph $G = (V, E, L)$ — vertices, edges, legs (possibly $\emptyset$)
- a vertex weight function $h: V(G) \longrightarrow \mathbb{Z}_{\geq 0}$
- an edge length function $d: E(G) \longrightarrow \mathbb{R}_{>0}$



genus $= 3 + 1 = 4$
$3 = \sum$ weights of vertices
$1 = b_1(G) = 1^{st}$ Betti # (G).

The genus of Γ is $g(\Gamma) = \sum_{v \in V} h(v) + b_1(G)$.

[Think: Γ = dual graphs of nodal curves : $h(v)$ = genus of the components, legs = markings

Automorphisms = iso of Γ that preserves both h & d.

• Γ is stable if $\forall v \in V \quad 2h(v) - 2 + val(v) > 0$

Non-examples: 

Def: The moduli space is $M_{g,n}^{trop} = \{[\Gamma] : \Gamma \text{ stable genus } g \text{ trop curve with } n \text{ marked legs}\}$
$2g - 2 + n > 0$

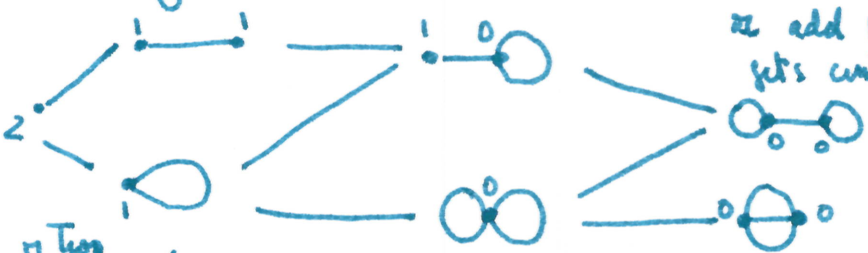
If no markings ⇝ M_g^{trop}

Prop: $M_g^{trop} = \bigsqcup_{\substack{\text{comb types} \\ G}} \mathbb{R}_{>0}^{E(G)} / \mathrm{Aut}(G)$

Gluing? $M_g^{trop} = \varinjlim_{G} \mathbb{R}_{\geq 0}^{E(G)} / \mathrm{Aut}(G)$

allow to contract edges, but need to add up weights of vertices & add 1 to vertex if a loop gets contracted

Eg M_2^{trop}



Poset of cones in M_2^{trop}

(can dilate the edge lengths globally)

Thm (Abramovich-Caporaso-Payne) $M_{g,n}^{trop}$ is a skeleton for $\mathcal{M}_{g,n}^{an}$.