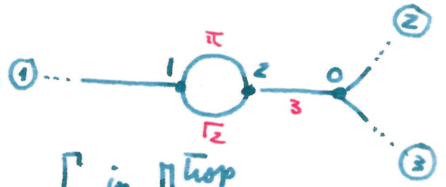
# Lecture XXXV: Moduli of tropical curves & curve counting



**Recall**: An *abstract tropical curve* $\Gamma$ is a triple consisting of:

- a finite graph $G = (V, E, L)$ (vertices, edges, legs)
- a vertex weight function $h: V(G) \to \mathbb{Z}_{\geq 0}$
- an edge length function $l: E(G) \to \mathbb{R}_{>0}$

Eg




$\Gamma$ in $M^{trop}_{g,n}$ $\quad g = 4 = 3+1$, $n = 3 = |L|$

$\Gamma$ is *stable* if $\forall v \in V \quad 2h(v) - 2 + \mathrm{val}(v) > 0 \rightsquigarrow$ modelled on stability on $\overline{M}_{g,n}$.

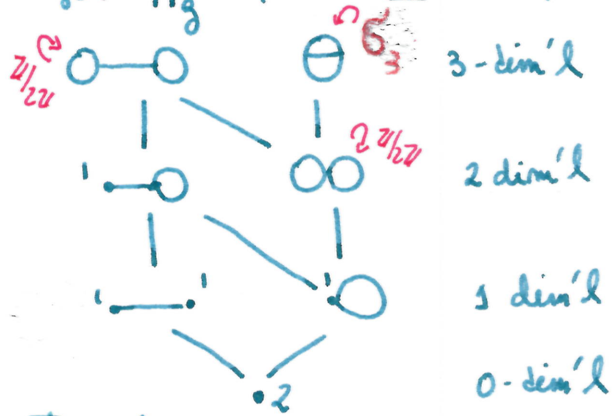
Genus of $\Gamma$: $g(\Gamma) = \sum_{v\in V} h(v) + \underbrace{b_1(G)}_{=|E|-|V|+1}$ ; Automorphisms of $\Gamma$: iso of $G$ preserving $h$ & $d$.

**Def** $M^{trop}_{g,n} = \{[\Gamma] : \Gamma$ stable genus $g$ trop curve with $n$ marked legs$\}$ $\quad 2g-2+n>0$

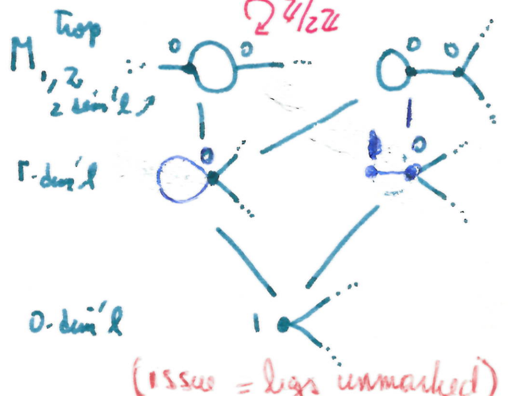
$\longleftrightarrow$ dual graphs of curves in $\overline{M}_{g,n}$

If $L = \emptyset$, we get $M^{trop}_g \rightsquigarrow$ not a tropical variety (not a balanced polyhedral fan)

Eg 1: $M^{trop}_2$




Eg 2 $M_{1,2}$ :

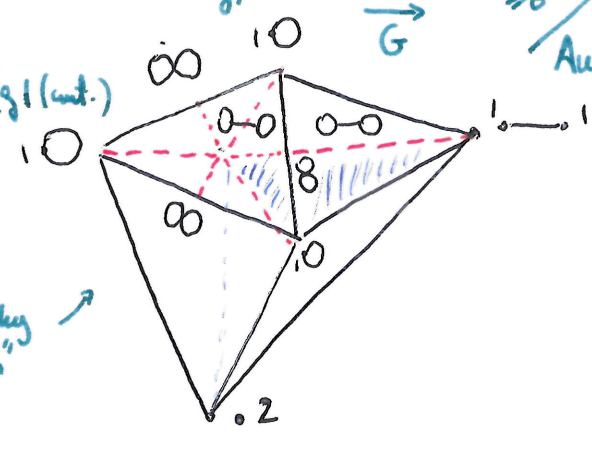



**Def** The *combinatorial type* of a curve $\Gamma$ is the data $(G, h)$, all but the edge lengths.
They correspond to dual graphs of curves classes in $\overline{M}_{g,n}/\mathfrak{S}_n$ (legs are unmarked)

**Prop**: $M^{trop}_{g,n} = \bigsqcup_{\substack{\text{comb types} \\ G}} \mathbb{R}^{E(G)}_{>0} / \mathrm{Aut}(G)$ & $\dim M^{trop}_{g,n} = 3g-3+n = \dim \overline{M}_{g,n}$

Gluing? $M^{trop}_{g,n} = \varinjlim_{G} \mathbb{R}^{E(G)}_{\geq 0} / \mathrm{Aut}\, G$ contraction of edges followed by


Eg 1 (cont.)

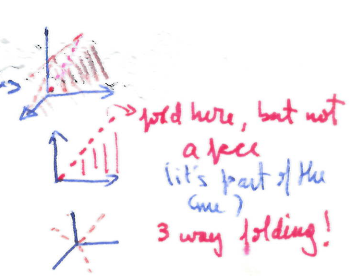



"stacky fan"

$G_1 =$ (graph) $\rightsquigarrow \mathbb{R}^3_{>0} / \mathbb{Z}/2\mathbb{Z}$

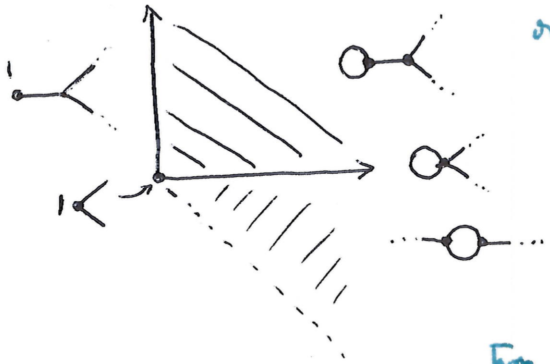
$G_2 =$ (graph) $\rightsquigarrow \mathbb{R}^2_{>0} / \mathbb{Z}/2\mathbb{Z}$

$G_3 =$ (graph) $\rightsquigarrow \mathbb{R}^3_{>0} / \mathfrak{S}_3$

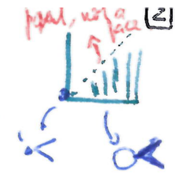



Rest = no automorphisms!

Eg 2 :



only one w/ autom $= \mathbb{R}^2_{>0} / \mathbb{Z}/2\mathbb{Z}$



**Thm** [Abramovich - Caporaso - Payne] $M^{trop}_{g,n}$ is a (Thuillier) skeleton for $\mathcal{M}^{an}_{g,n}$.

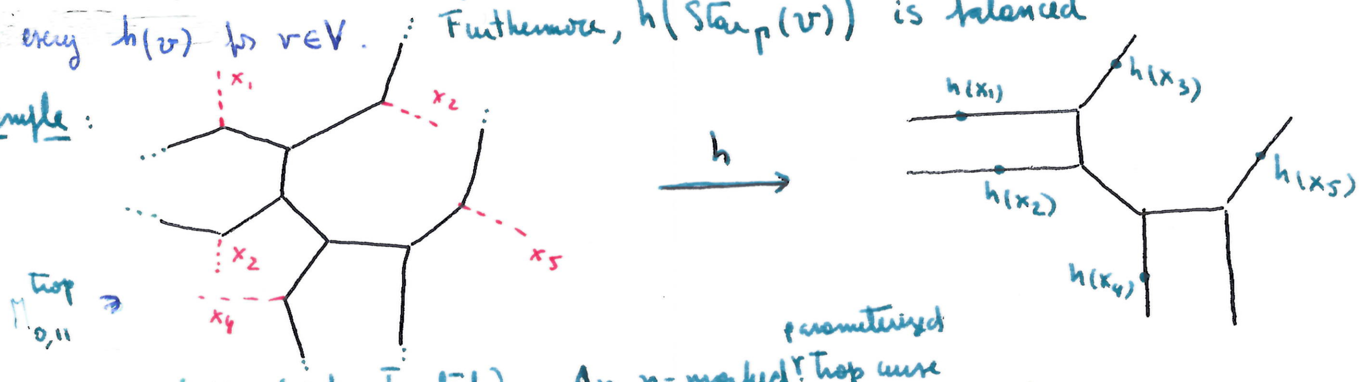
§2 Parameterized tropical curves : ignore weights on vertices.

- Each abstract tropical curve $\Gamma$ is a topological space = union of line segments (half rays & legs or bounded edges)
- Why tropical? Balancing not part of the definition!

**Def** An n-marked parameterized tropical curve $\Gamma$ on $\mathbb{R}^m$ is $h : (\Gamma, x_1, \ldots, x_n) \to \mathbb{R}^m$ satisfying

- $h$ is a continuous map ; $x_1, \ldots, x_n$ are some legs of $\Gamma$
- $h$ is affine linear on each edge of $\Gamma$ with rational slope. & $h|_E : [0, \ell(E)] \to \mathbb{R}^2$, $t \longmapsto a + v(E) t$, $v(E) \in \mathbb{Z}^2$
- $h(v) \in V(h(\Gamma))$ $\forall v \in V$ , $h$ contracts the legs $x_1, \ldots, x_n$
- We can equip the edges of $h(\Gamma)$ with integer weights so that $h(\Gamma)$ is balanced at every $h(v)$ for $v \in V$. Furthermore, $h(\mathrm{Star}_\Gamma(v))$ is balanced

Example :




(all $x_i$'s get contracted) An n-marked parameterized trop curve

**Def**: The degree of a marked plane trop curve $(\Gamma, h)$ is defined as the unordered tuple $\Delta = \{ h(e) - h(v_e)$ for $e \in L$, $v$ vertex of $e$, $h(e) \neq pt \}$

Motivation: For $m = 2$, $\Delta$ will give the rays of the fan of a toric surface $\Sigma$ containing a plane curve $C$ whose tropicalization is $h(\Gamma)$.

Eg above : $\Delta = \{ \begin{bmatrix} -1 \\ 0 \end{bmatrix} \times 2, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \times 2, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \times 2 \}$ $C$ degree 2 curve in $\mathbb{P}^2$.

**Def**: Isomorphisms of n-marked trop curves $(\Gamma, x_1, \ldots, x_n, h)$ & $(\Gamma', x'_1, \ldots, x'_n, h')$ if $\exists$ iso $\varphi : (\Gamma, x_1, \ldots, x_n) \longrightarrow (\Gamma', x'_1, \ldots, x'_n)$ st $h' \circ \varphi = h$