Lecture XXXVI: Curve counting in tropical geometry

Idea: Choose some geometric object (eg curves) satisfying certain conditions (eg passing through certain number of points, or with certain tangency conditions at ∞), and count them. (Impose conditions so that the number of such objects is finite)

Eg: How many irreducible nodal curves in $\mathbb{P}^2$ of degree d & genus g passing through $3d+g-1$ general points are there? legs $= n+3d$ (for the abstract trop curve)

If the points are general, the count is finite $=: N_{g,d}$ Q: Why $3d+g-1$ pts? (page 2)

Eg $N_{0,1} = 1$ $\quad 3d+g-1 = 2$ pts

$N_{0,2} = 1$ $\quad 3d+g-1 = 5$ pts

$N_{0,3} = 12$ $\quad 3d+g-1 = 8$ pts

$$g = \underbrace{\tfrac{1}{2}(d-1)(d-2)}_{= \#\text{ interior lattice points of } d\Delta_2 \text{ in } \mathbb{Z}^3} - \#\text{ nodes}$$

$\hookrightarrow$ $\frac{1}{2}(d-1)(d-2) = 1$ so C has a single node $\Rightarrow$ Condition: discriminant of a cubic must vanish, and this is a deg 12 polynomial.

Theorem [Mikhalkin's Correspondence 2004] $N_{g,d}$ can be computed tropically & it equals $N^{trop}_{g,d}$.

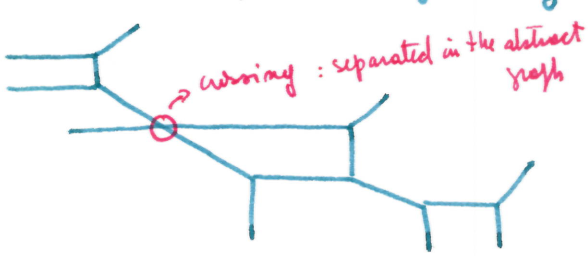
Here, $N^{trop}_{g,d}$ = # of simple ~~nodal~~ trop curves in $\mathbb{TP}^2$ of degree d & genus g passing through $3d+g-1$ tropically general points in $\mathbb{R}^2$, counted with multiplicity

Furthermore = multiplicities have a combinatorial formula = # algebraic curves w/ given tropicalization.

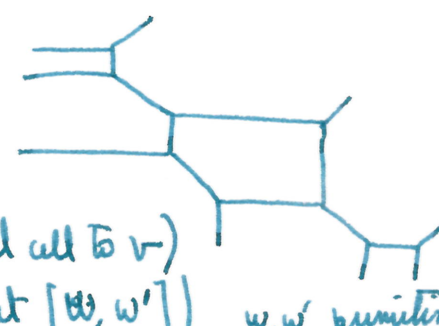
Tropical translation (see (**))  

- simple nodal trop curve: only 3- & 4-valent vertices ("crossings") edges that cross but don't meet.
  - degree d = dual to a subdivision of $d\Delta_2 = \text{convex hull}((0,0),(d,0),(0,d))$
  - genus g = the graph has genus g (4-valent vertices are crossings)

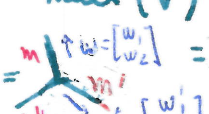
Eg $g=0$: $d=3$



vs $g=1$: $d=3$



$$\text{Mult}(\Gamma) = \prod_{\substack{v\in V \\ \text{val}(v)=3}} \text{mult}(v) \qquad \text{mult}(v) = 2(\text{area dual cell to } v)$$



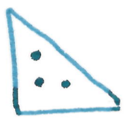
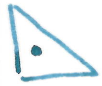
$v = $ $= m_{trop}(w)\, m_{trop}(w') \left| \det [w, w'] \right|$ $\quad w, w'$ primitive vectors

Balancing $\Rightarrow$ independent of pair chosen!

(*) Q: Why $3d+g-1$ pts?

Plane Algebraic Geometry (18th Century)

If C is a smooth genus curve of degree d in $\mathbb{P}^2_{\mathbb{C}}$, then its genus is
$g = \frac{1}{2}(d-1)(d-2) = \#\left(\mathbb{Z}^2 \cap \text{interior}(\text{Newton Polytope}(C))\right)$

Eg: $d=4$, $g=3$  $d=3$, $g=1$  (elliptic)

The genus drops by 1 for each nodal singularity.

The Projective space of curves of degree d has dimension $\binom{d+2}{2} - 1 = \underbrace{\tfrac{1}{2}(d-1)(d-2)}_{g} + \underbrace{3d}_{\text{correction term}} - 1$

monomials of degree d in x, y, z ; global scalar.

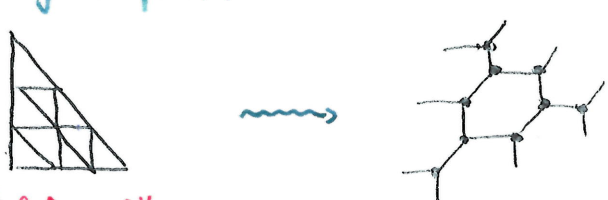
Points in general: $3d-1+g$ independent conditions so we expect a dim $=0$ space.

(**) Q: Why tropical translation works?

Def: A plane tropical curve is smooth of degree d if it is dual to a unimodular triangulation of $d\Delta_2 = \text{conv hull}\{(0,0),(0,d),(0,d)\}$ $\rightsquigarrow$ so $b_1(\Gamma)$ is maximal $= \frac{1}{2}(d-1)(d-2)$.

Unimodular = each Δ has volumen 1 (normalized!)

Eg: honeycomb form:



$t = 9\,\Delta \leftrightarrow 9V$

$r = \#$ unbounded edges $= 3 \cdot$ degree

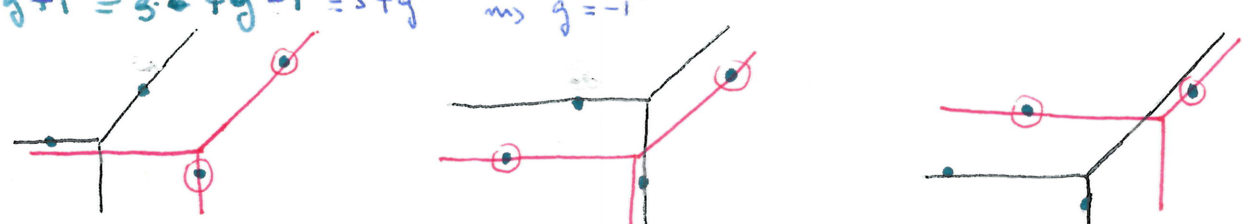
$b_1(\Gamma) = |E| - |V| + 1 = \#$ bounded regions.
$= 6 - 6 + 1$

The genus of a simple curve is $\frac{1}{2}t - \frac{1}{2}r + 1$

Example 1: How many singular geodesic trop curves in the plane pass through 4 given pts? → union of 2 lines

$4 = 3d + g - 1 = 3\cdot 2 + g - 1 = 5 + g$ $\rightsquigarrow$ "$g = -1$"

$A = 3$



Reason: choose a pair to get a line so $\binom{4}{2}/2 = 6/2 = 3$ → Symmetry!

Each one has mult $= 1$

$N^{\text{nd}}_{-1,2} = 3$

$\rightsquigarrow g = -1$ (no interior lattice pt!)

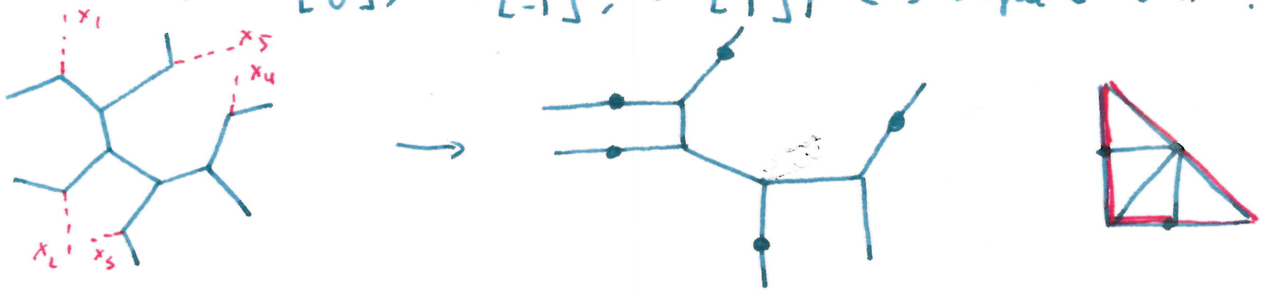
Example 2 $N_{0,3} = 12$ $3d+g-1 = 8$ (Pick all 8 points on a line with slope $-\varepsilon$, $\varepsilon \in (0,1)$, $\varepsilon \notin \mathbb{Q}$)

Combinatorial count? Lattice paths! $N_{path}(d\Delta_2, g) = N^{trop}_{plane}(g,d)$ [Mikhalkin]

$n = 3d+g-1$ = marked points of an n-marked param. trop curve $(\Gamma, x_1, \dots, x_n, h)$

with $\Delta = \{ d \times \begin{bmatrix} -1 \\ 0 \end{bmatrix}, d \times \begin{bmatrix} 0 \\ -1 \end{bmatrix}, d \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} \} \longleftrightarrow$ degree d in $\mathbb{P}^2$.   → Lecture XXXV

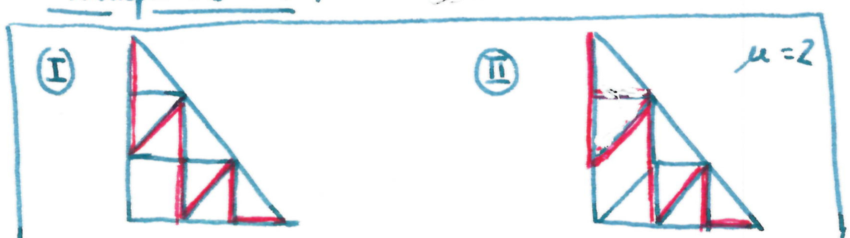
Eg $d=2$, $g=0$, $n=5$



dual

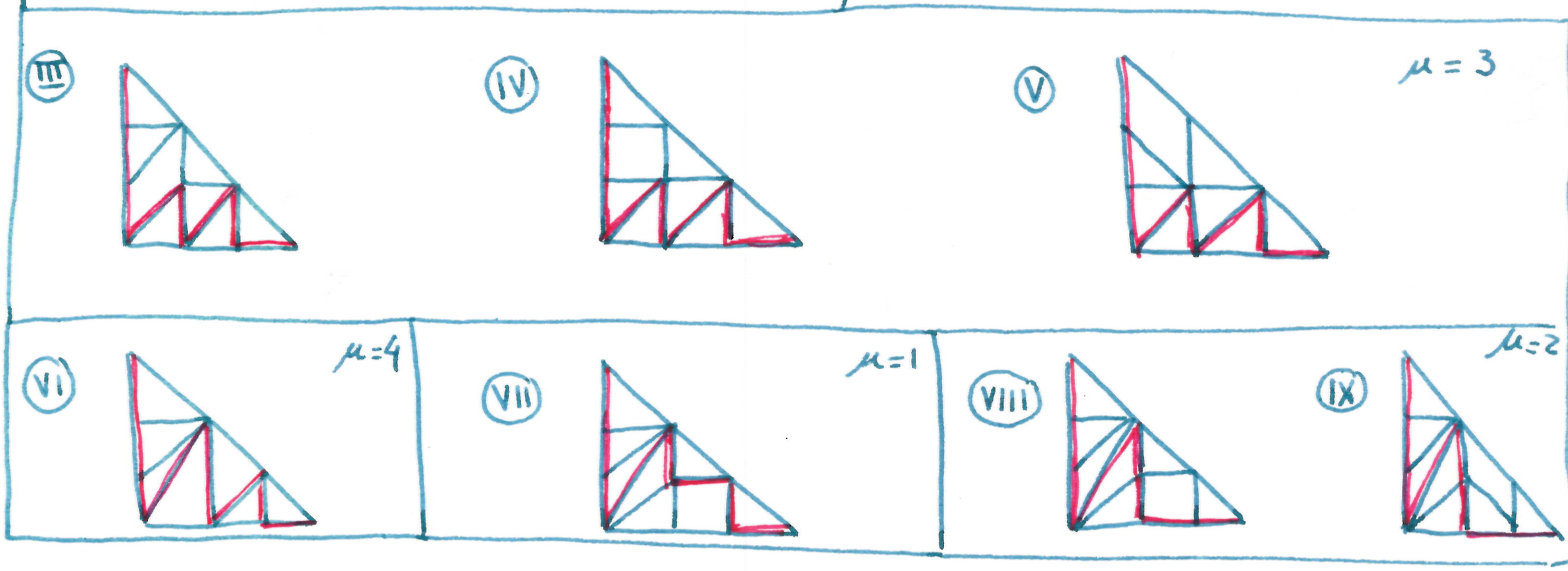
Marked dual subdivision of $d\Delta_2$ = mark the edges dual to edges in Trop curve $h(\Gamma)$ containing each $h(x_1), \dots, h(x_n)$

Example 2 (cont):



→ see page 5

$2+3+4+1+2 = 12$



Marked edges in the subdivision define a lattice path Γ (λ-increasing) in $d\Delta_2$ where $\lambda(x,y) = x - \varepsilon y$ ($\lambda : \mathbb{R}^2 \to \mathbb{R}$ linear with kernel of irrat'l slope)

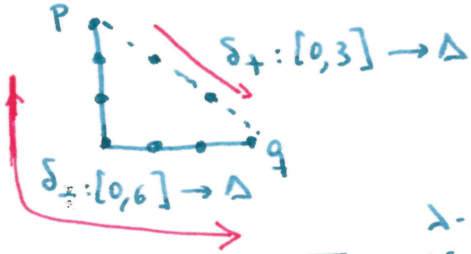
Def: • A path $\gamma : [0,n] \longrightarrow \mathbb{R}^2$ is a lattice path if $\gamma|_{[j-1,j]}$ is affine linear $\forall j = 1, \dots, n$ and $\gamma(j) \in \mathbb{Z}^2$ $\forall j = 0, \dots, n$

• Given $\lambda : \mathbb{R}^2 \longrightarrow \mathbb{R}$ linear with kernel of irrational slope (eg $\lambda(x,y) = x - \varepsilon y$, $0 < \varepsilon \ll 1$, $\varepsilon \notin \mathbb{Q}$) we say γ is λ-increasing is $\lambda \circ \gamma$ is strictly increasing.

• Take p, q in $\partial\Delta = \partial\Delta_2$ where λ reaches the min & max values. This divides $\partial\Delta$ into 2 λ-increasing lattice paths:

$$\delta_+ : [0, n_+] \longrightarrow \partial\Delta \qquad (\text{clockwise around } \partial\Delta)$$

$$\delta_- : [0, n_-] \longrightarrow \partial\Delta \qquad (\text{counterclockwise around } \partial\Delta)$$

Eg: 

$N_{path}(\partial\Delta_2, g) := \#(\lambda\text{-incr. lattice paths } \gamma : [0, n] \longrightarrow \partial\Delta_2,\ \gamma(0) = p,\ \gamma(n) = q)$ (counted with multiplicity)

• Define a multiplicity for λ-increasing lattice paths recursively:

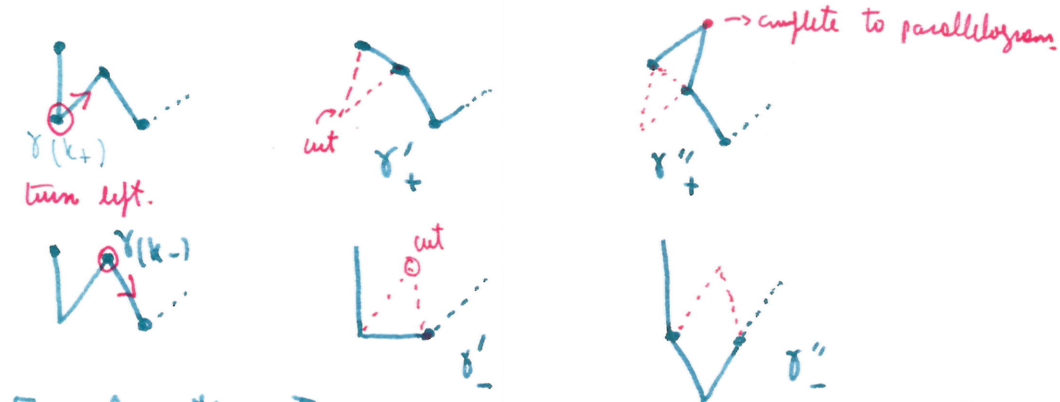
(1) $\mu_\pm(\delta_\pm) = 1$

(2) If $\gamma \neq \delta_\pm$, pick $k_\pm \in [0, n]$ smallest number where γ makes a LEFT turn (right turn for μ_-) [If no such $k_\pm$ exist, set $\mu_\pm(\gamma) = 0$]

From here, define 2 more λ-increasing paths γ' & γ'':

• $\gamma'_\pm : [0, n-1] \longrightarrow \Delta$ path that cuts the corner of $\gamma(k_\pm)$, ie
$\gamma'_\pm(j) := \gamma(j)$ for $j < k_\pm$, $\quad \gamma'_\pm(j) := \gamma(j+1)$ for $j \geq k_\pm$.

• $\gamma''_\pm : [0, n] \longrightarrow \Delta$ path that completes the corner of $\gamma(k_\pm)$ to a parallelogram, ie $\gamma''_\pm(j) := \gamma(j) \ \forall j \neq k_\pm$, $\gamma''_\pm(k_\pm) = \gamma(k_\pm - 1) + \gamma(k_\pm + 1) - \gamma(k_\pm)$

Idea: 

• Pick T = triangle with vertices $\gamma(k_\pm - 1), \gamma(k_\pm), \gamma(k_\pm + 1)$, then

$$\mu_\pm(\gamma) = 2\,\text{Area}\,T\ \mu_\pm(\gamma'_\pm) + \mu_\pm(\gamma''_\pm)$$

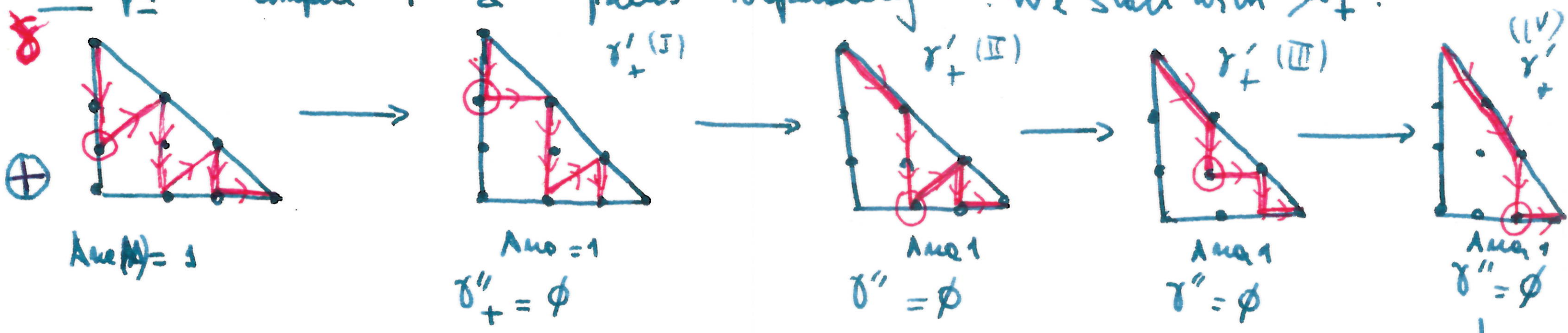
• both include smaller area with $\delta_\pm$ $\Rightarrow$ induct!

• Mult $\boxed{\mu(\gamma) = \mu_+(\gamma)\,\mu_-(\gamma)}$

• 0 if $\gamma'_\pm$ (resp $\gamma''_\pm$) leave Δ ~~n~~ if only left / right turns

Note: Only end paths that don't count 0 are δ_+ & δ_- (with the corresponding speed)

Example: Compute + & − paths separately. We start with μ_+:

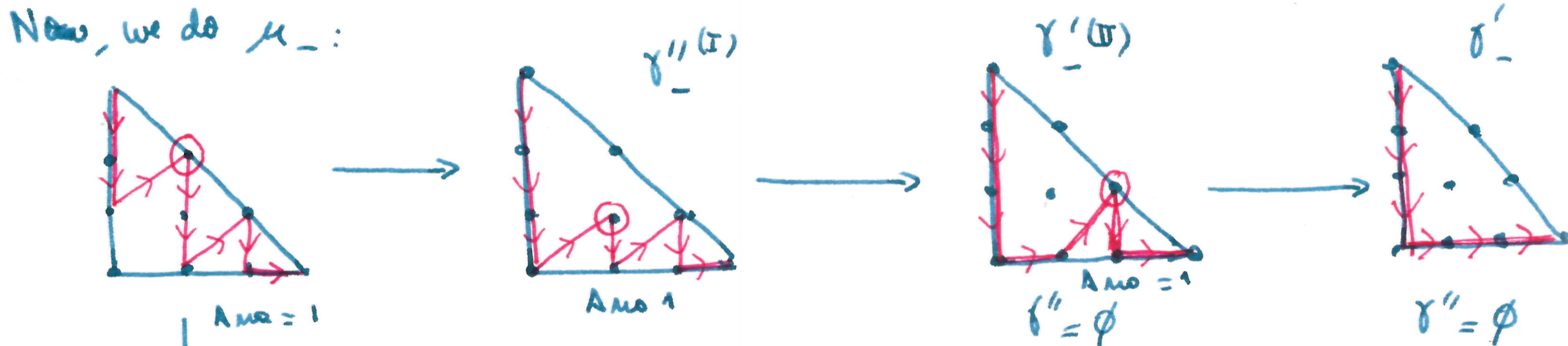


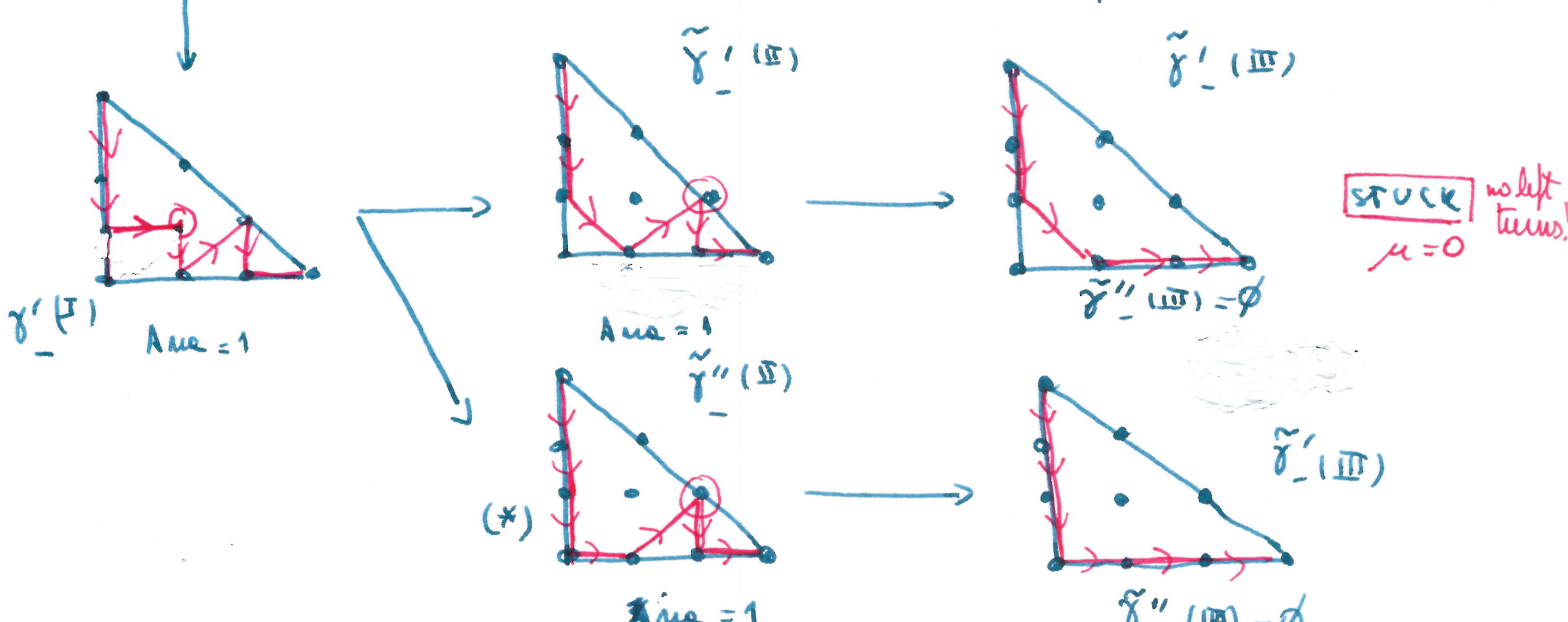
$\mu = 1$ $\gamma' = \delta_+$
$\gamma'' = \emptyset$

Get last multiplicity & trace back $\mu(\gamma'_+) = 1 \cdot 1 \cdot 1 \cdot 1\, (\mu(\delta_+)) = \boxed{1}$

So $\mu_+(\gamma) = 1 \cdot \mu(\gamma'_+) + 0 = \boxed{1}$.

Now, we do μ_-:





$$\mu_-(\gamma''_-(\mathrm{I})) = 1\,(1\,\underbrace{\mu(\gamma'_-)}_{=1}) = 1.$$

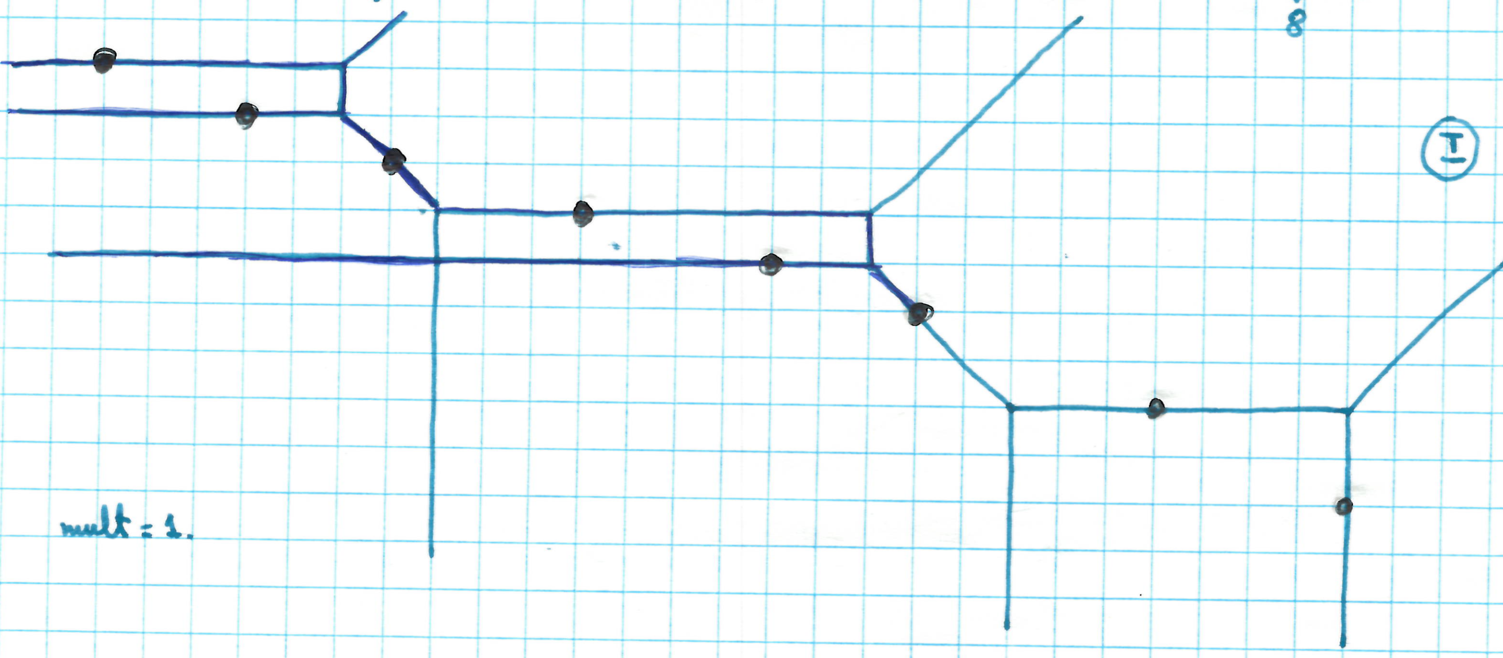
$$\mu_-(\gamma'^{(\mathrm{I})}_-) = 1 \cdot 0 + \mu_+((*)) = 1.$$

$$\Rightarrow \mu_-(\gamma) = 1 \cdot \mu_-(\gamma''^{(\mathrm{I})}_-) + \mu_-(\gamma'^{(\mathrm{I})}_-) = 1 \cdot 1 + 1 = \boxed{2}$$

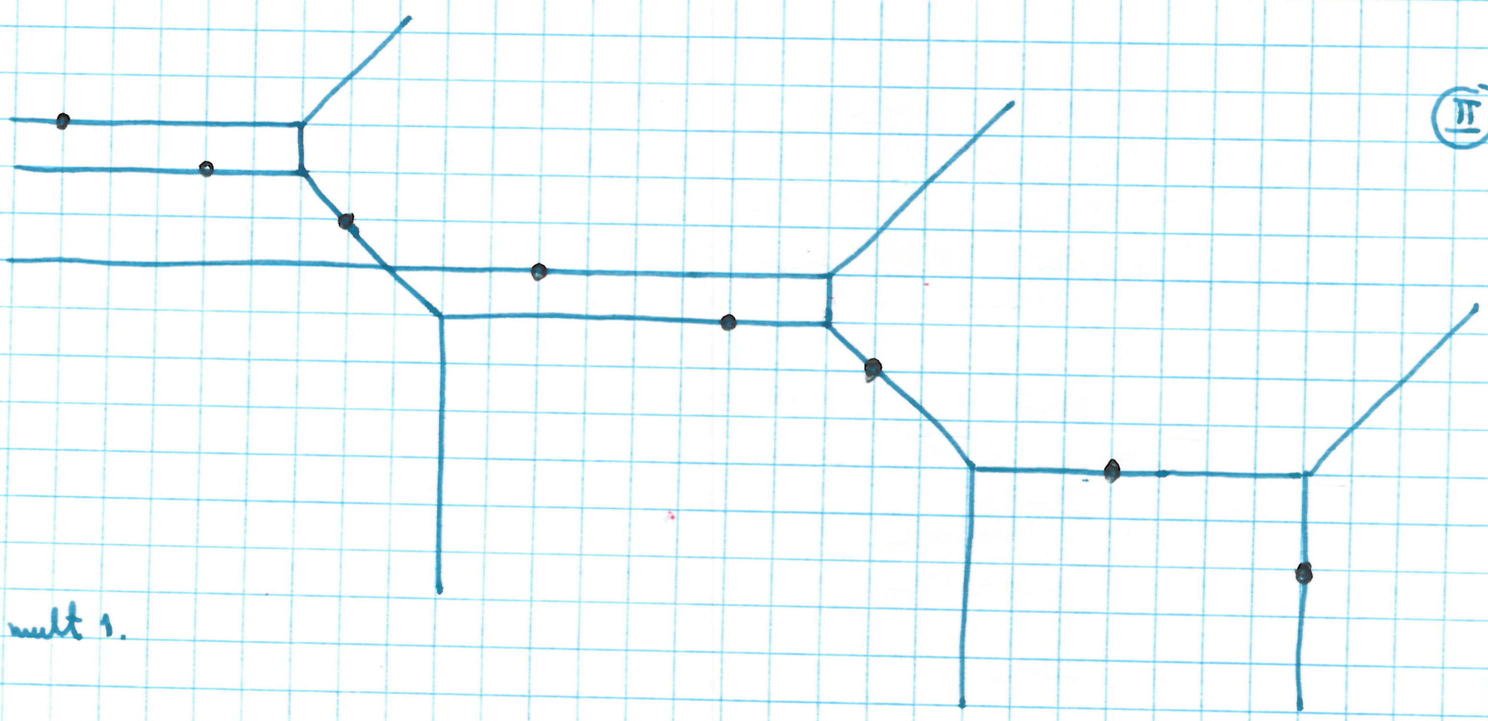
Conclusion:

$$\boxed{\begin{aligned}\mu &= 1 \cdot 2\\ &= 2\end{aligned}}$$

Example : $N_{0,3} = 12$ $\quad 3d+g-1 = 8 \quad$ & $\quad 12 = \underbrace{1+1+\cdots+1}_{8} + 4\cdot 1$

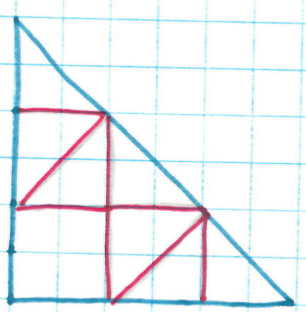


mult = 1.

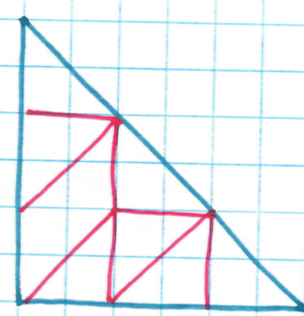


mult 1.

I



II



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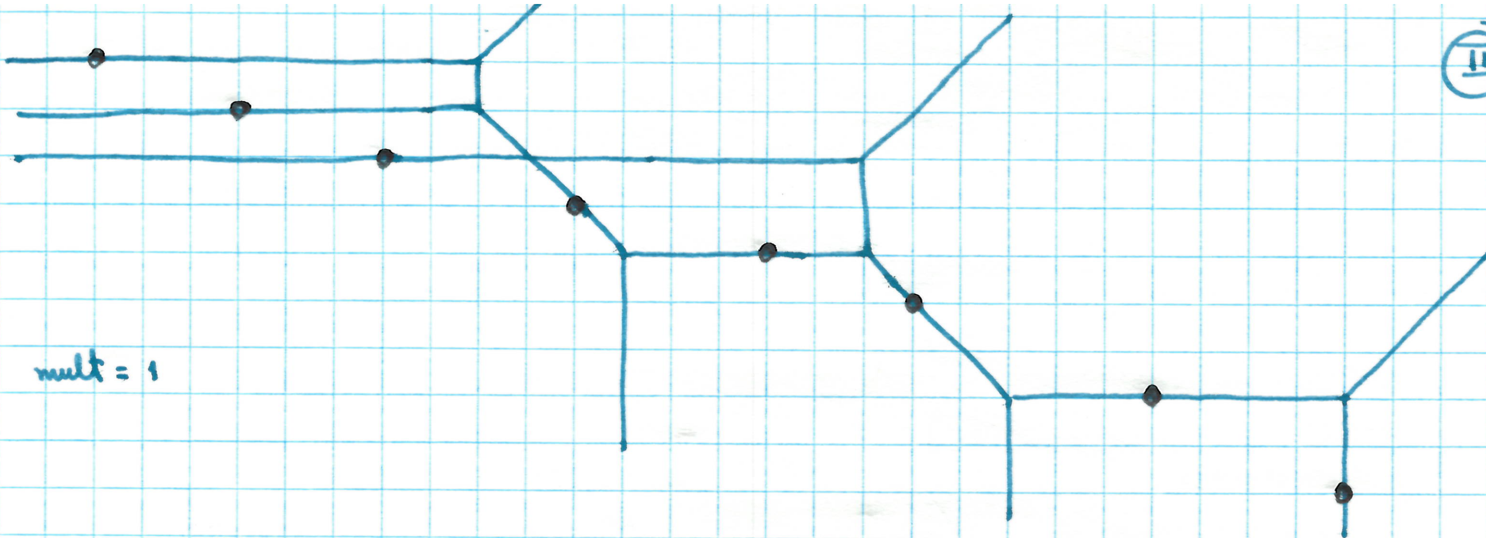
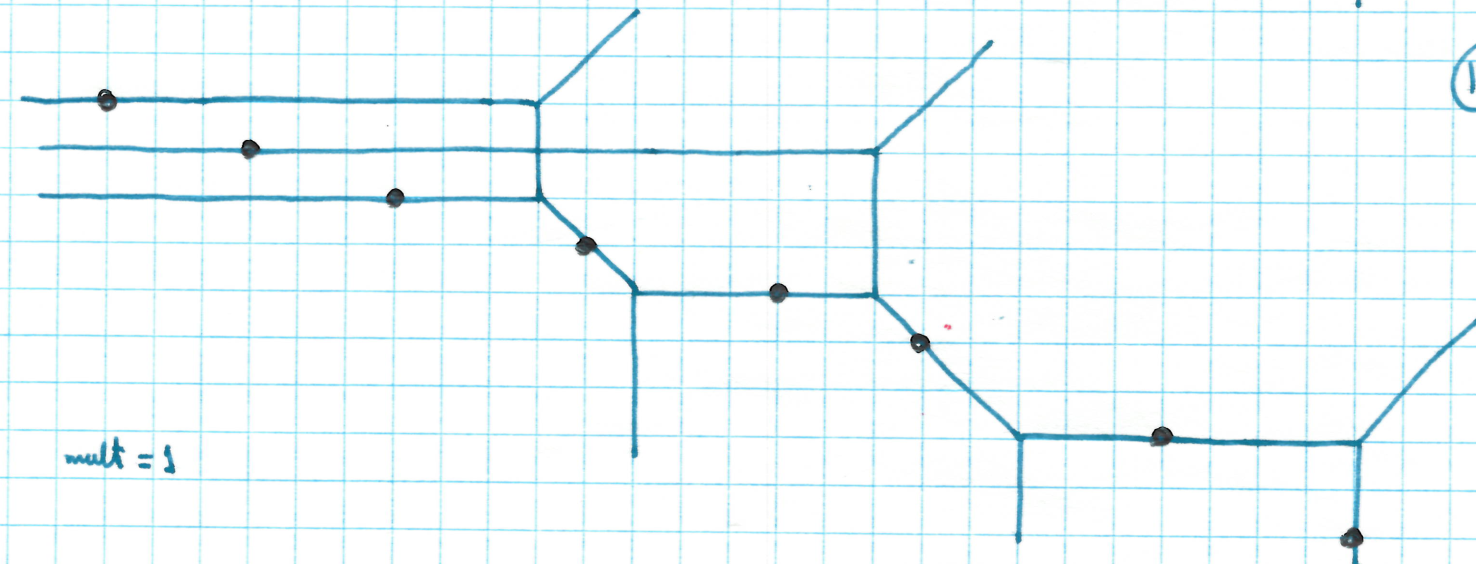


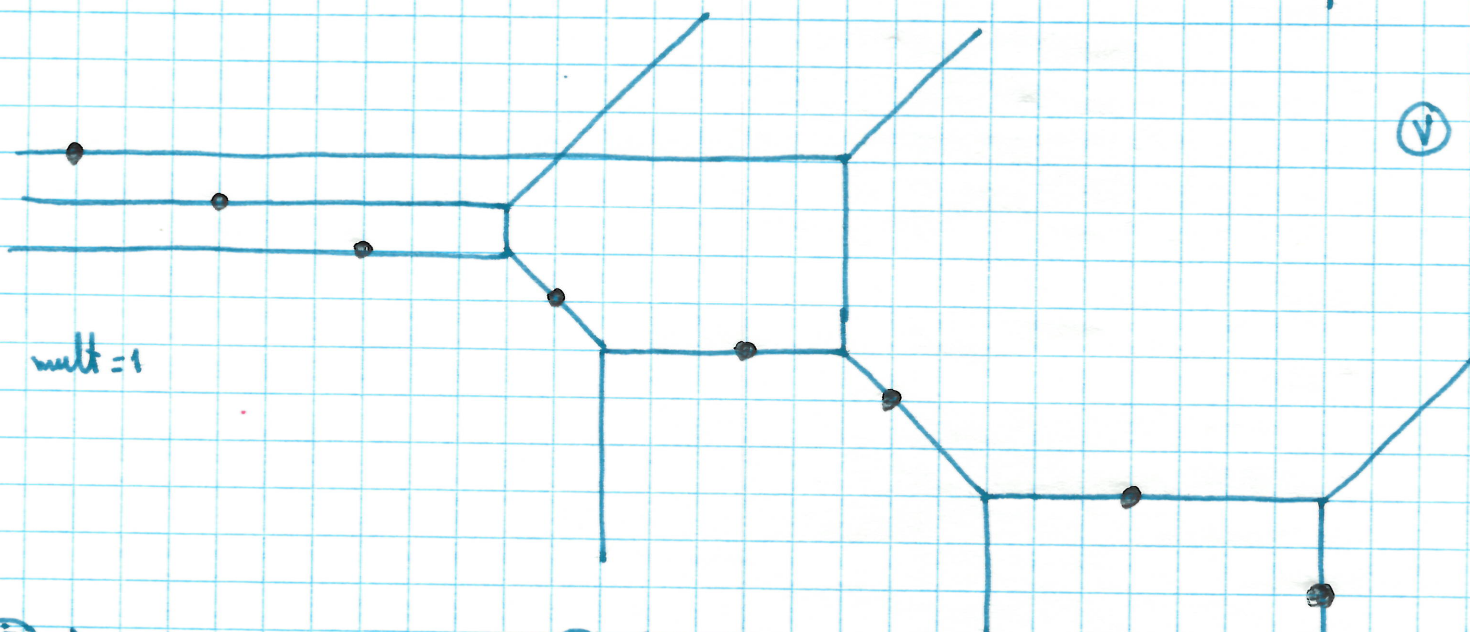
mult = 1

IV



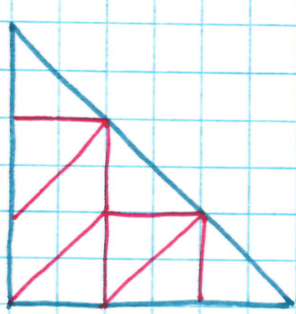
mult = 1

V

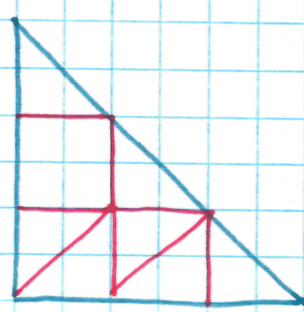


mult = 1

III



IV



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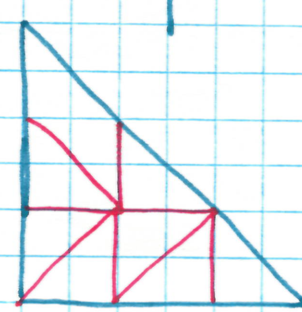
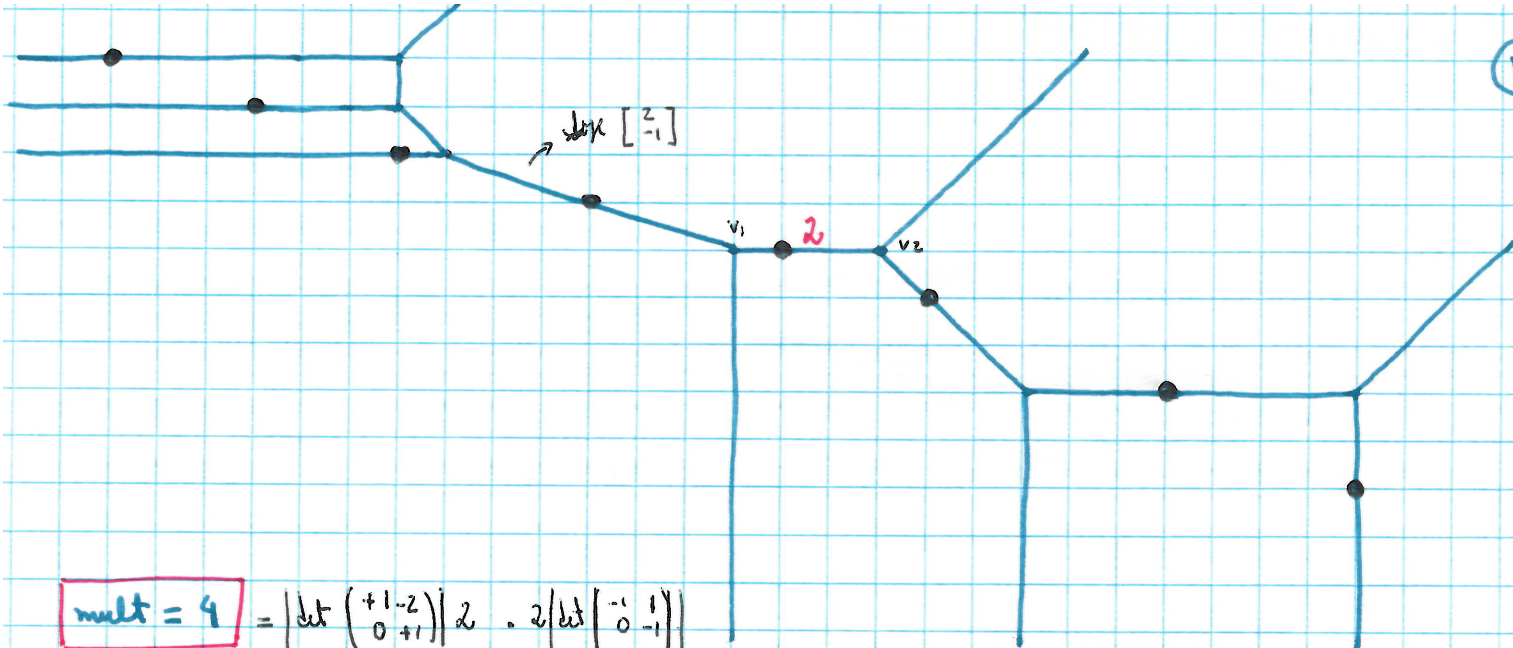
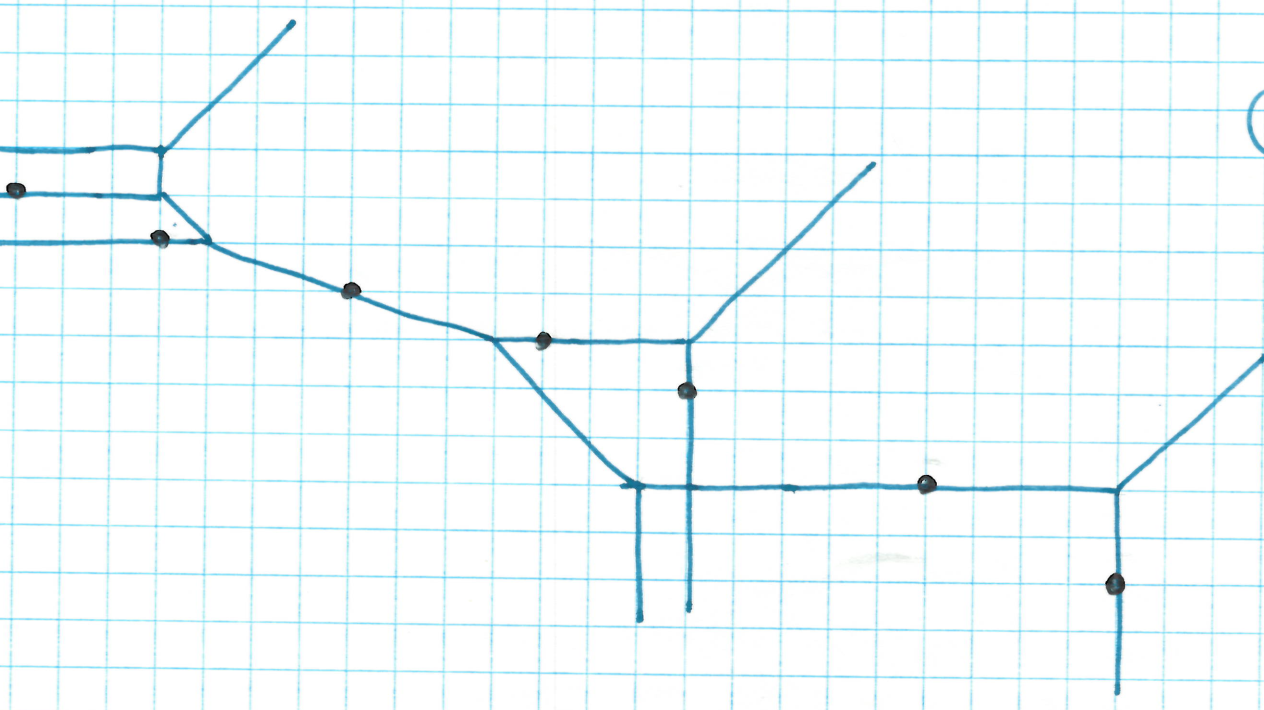


VI



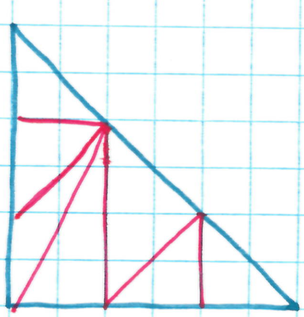
$$\boxed{\text{mult} = 4} = \underbrace{\left|\det\begin{pmatrix}+1 & -2\\ 0 & +1\end{pmatrix}\right| 2}_{\text{mult}(v_1)} \cdot \underbrace{2\left|\det\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix}\right|}_{\text{mult}(v_2)}$$

VII

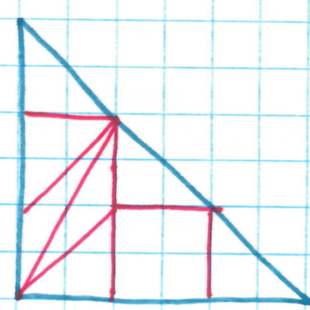


mult = 1

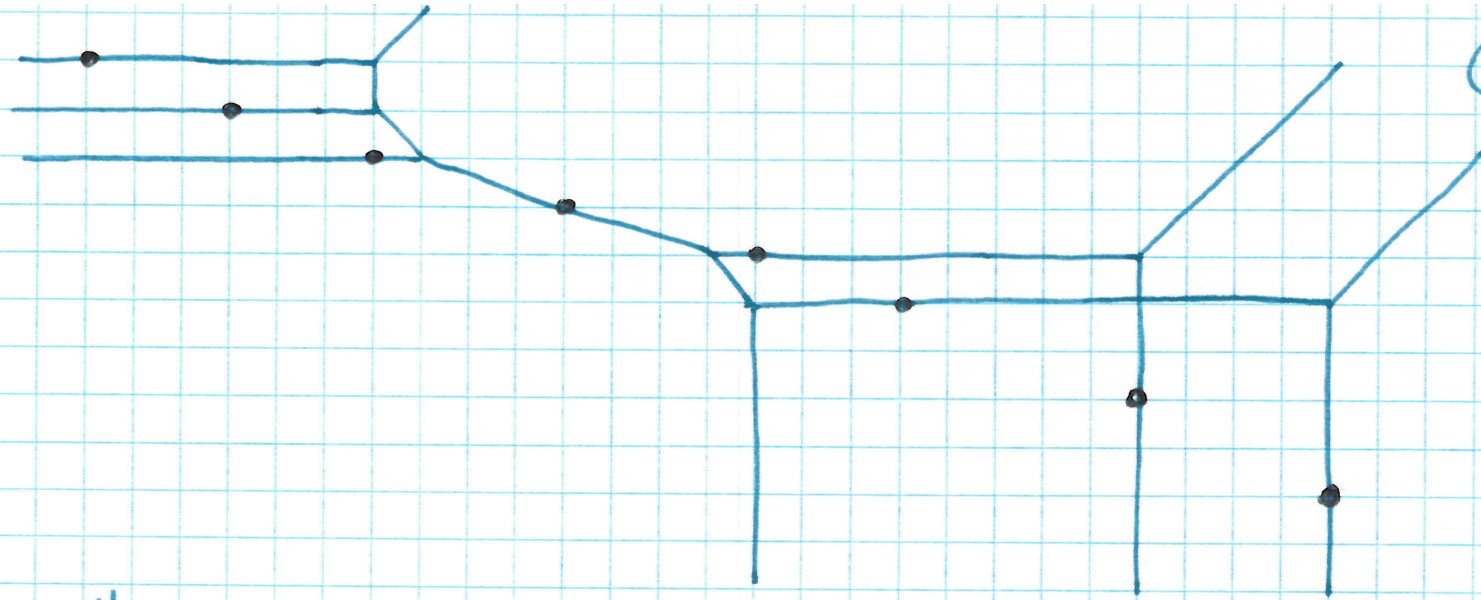
VI



VII

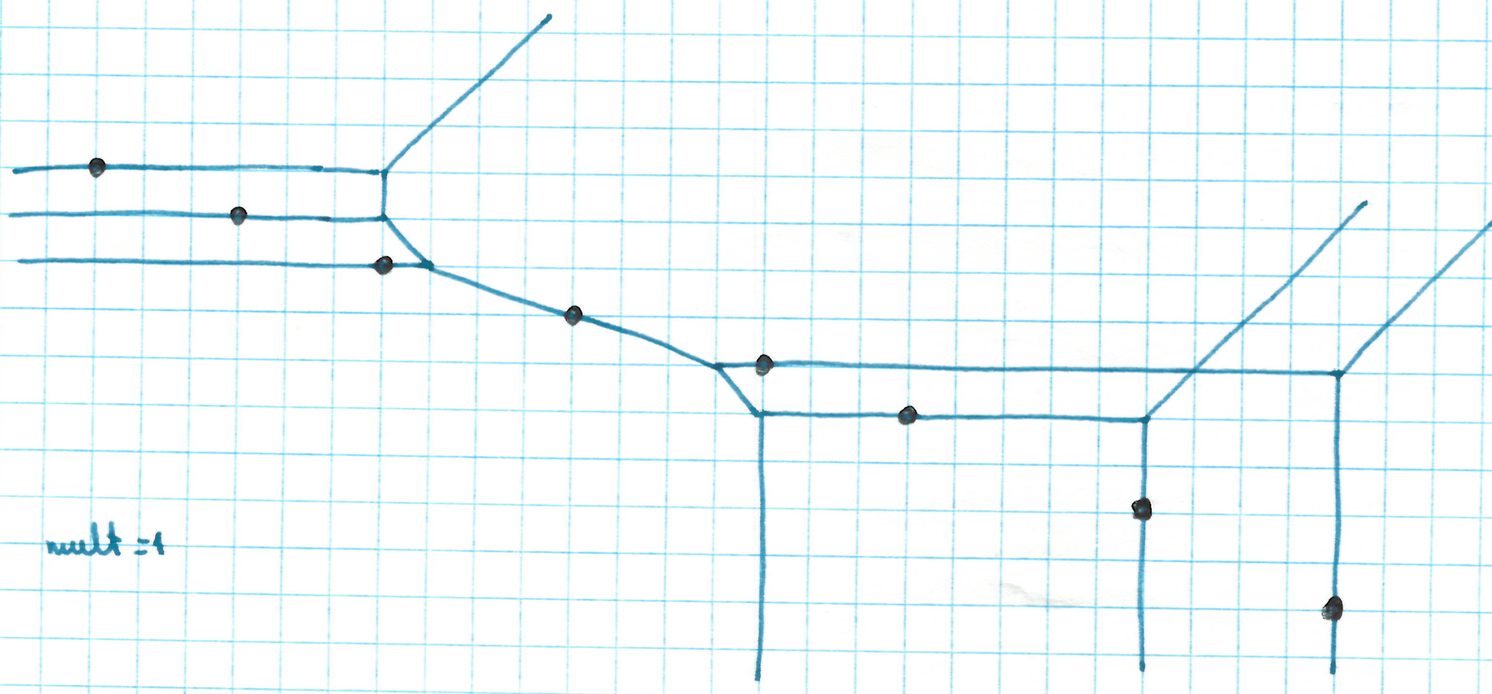


VIII



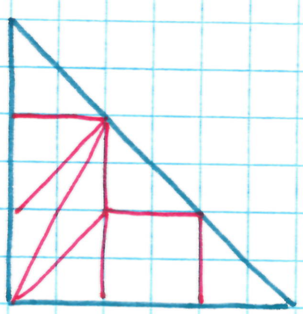
mult = 1

IX



mult = 1

VIII



IX