Lecture XXXVIII: Tropical Geometry of genus 2 curves

§1 Warm-up: genus 1

Recall: Smooth genus 1 curves $E/\mathbb{C}$ are given by $y^2 = x(x-1)(x-\lambda)$, for $\lambda \neq 0, 1, \infty$. So $E \xrightarrow{2:1} \mathbb{P}^1$ cover branched at 4 points: $0, 1, \infty, \lambda$ in $\mathbb{P}^1$

$\rightsquigarrow$ $j(E) = 256 \dfrac{(1-\lambda-\lambda^2)^3}{\lambda^2(1-\lambda)^2}$ characterize isomorphism classes of Elliptic curves

2 values of j are special $j = 0, 1728$ (extra automorphisms)

Can extend to field K of char. 0, eg $K = \mathbb{C}\{\{t\}\}$.

GW-Theory: counts # elliptic curves subject to certain conditions.

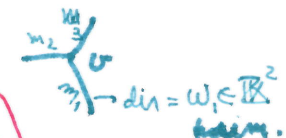
$E(d, j) = \#\{$elliptic curves E of degree d through $3d-1$ pts $w/ j = j(E)\}$

Thm [Pandharipande '97] $E(d,j) = \binom{d-1}{2} N_{0,d}$ for $j \neq 0, 1728$, with an extra factor counting extra automorphisms for $j = 0, 1728$.

Thm [Kerber-Markwig '09] $E_{trop}(d,j) := \#\{$simple trop curves Γ of genus 1 & degree d in $\mathbb{TP}^2$ with prescribed tropical j-invariant passing through $(3d-1)$ trop real pts$\}$

when each Γ is counted with $\text{mult}(\Gamma) = \prod_{\substack{v \in V(\Gamma) \\ \deg v = 3}} \text{mult}(v)$

$\text{mult}(v) = m_1 m_2 |\det(w_1, w_2)|$ [index of device by the balancing condition] "à la Mikhalkin"



. trop j-invariant $=$ $\mathbb{Z}$-length of the unique loop in Γ $\in \mathbb{R}_{\geq 0}$.

. genus $= b_1(\Gamma) = |E| - |V| + 1$

Then $E_{trop}(d,j) = \binom{d-1}{2} N^{trop}_{0,d}$ $\forall j \in \mathbb{R}_{\geq 0}$ & it's independent of j.

- Len-Ranganathan ['16] Extend these results to elliptic curves on other toric surfaces & count non-realizable elliptic curves (superabundance phenomenon)

Motivation: trop-j invariant $= -\text{val}(j_E)$ for $E \subseteq \mathbb{P}^2$ sm cubic with bad reduction.

" length (skeleton of E^{an})

Questions: (1) What happens for $g > 1$? Many more combinatorial types!

(2) What plays the role of j_E?

Before attempting any GW Theory, need to study the combinatorics! Start with genus 2.

§2 Genus 2 Trop curves

Genus 2 curves have a poset w/ 7 cells ⇒ combinatorics is more delicate!

But, we can describe them by an hyperelliptic equation $y^2 = f(x)$ (over char $K \neq 2, 3$) where f has degree $n = 2g+2$ & has n distinct roots.

This gives a description as a hyperelliptic curve (for $g=2$, $n=6$)

In general, a genus g curve X is hyperelliptic if it admits a hyperelliptic equation. Equivalently: there exists

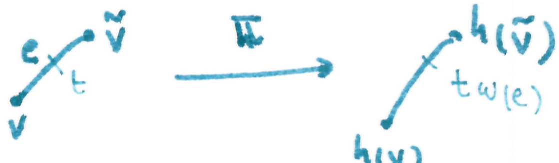
$$C \xrightarrow{2:1} \mathbb{P}^1$$

cover branched at $n = 2g+2$ points ($\longleftrightarrow$ roots of the hyperelliptic equation)

Q: Tropical analogs of this?

Def: A cover of an abstract tropical curve Γ' by an abstract trop curve Γ is a map $\pi : \Gamma \longrightarrow \Gamma'$ that is a surjective harmonic map of metric graphs. That is satisfying the local Riemann-Hurwitz conditions at each point.

- π is piecewise affine linear with integer slopes:
  - $V(\Gamma) \xrightarrow{\pi} V(\Gamma')$
  - e edge of Γ of length $l(e) \in \mathbb{R}_{>0}$, then $e' = \pi(e)$ is an edge of Γ' of length $l(e')$ & $\pi|_e : [0, l(e)] \longrightarrow [0, l(e')]$, $t \longmapsto \omega(e)t$ with $\omega(e) \in \mathbb{Z}$. (speed of π along e)

Eg 

so $\boxed{\omega(e) = \dfrac{l(e')}{l(e)}}$ slope of $\pi|_e$

- harmonic = balanced at every vertex $v \in V(\Gamma)$

For any edge $e' \ni v' = \pi(v)$ in Γ', then $d_v := \sum_{\substack{e \in E(\Gamma) \\ \pi(e) = e'}} \omega(e)$ is independent of e'. We call d_v the local degree of π at v.

(Can extend this notion to internal points of edges by $d_v = \omega(e)$ if $v \in \operatorname{rel\,int}(e)$).

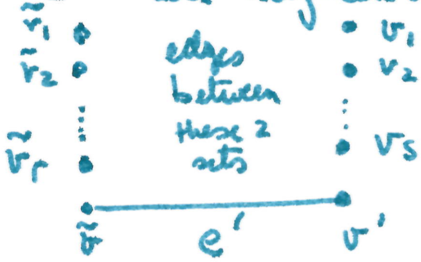
Local Riemann Hurwitz condition at $v \in V(\Gamma)$:

$$2-2g(v) = d_v\,(2-2g(v')) - \sum_{\substack{e \in E(\Gamma) \cup L \\ v \in e \\ [\ell(e) \leq \infty]}} (w(e)-1).$$

Def: The degree of a cover π is $d = \sum_{\pi(v)=v'} d_v$ for any fixed $v' \in V(\Gamma')$

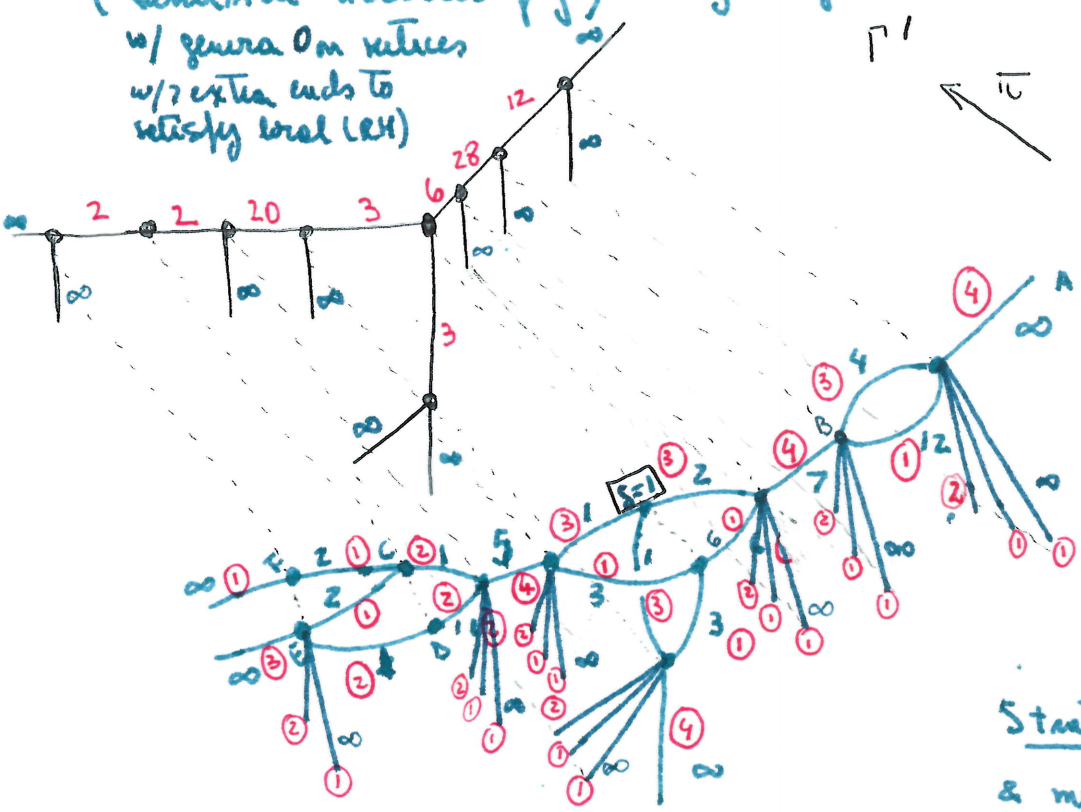
Note: This number is independent of v' by the balancing condition (harmonicity)

Why? The graphs Γ & Γ' are connected & d from v' & d from $\tilde{v}'$ agree whenever v' & $\tilde{v}'$ are adjacent vertices in Γ'.



$$\Rightarrow \text{So} \sum_{i=1}^{r} \underbrace{\sum_{\substack{e \text{ over } e' \\ \tilde{v}_i \in e}} w(e)}_{d_{\tilde{v}_i}} = \sum_{i=1}^{s} \underbrace{\sum_{\substack{e \text{ over } e' \\ v_i \in e}} w(e)}_{d_{v_i}}$$

Example: We can construct a degree 4 cover of the tropical line Γ' ($g=0$) (subdivide accordingly) by a genus 5 metric graph Γ

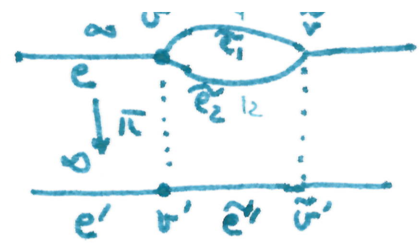


All vertices in Γ have genus 0 with one exception.

Need to fill in the weights on Γ' & lengths on Γ so that we get a degree 4 cover $\pi : \Gamma \to \Gamma'$

Strategy: start from 1 end (A) & move within the graph to compute
- weights for π
- local degrees
- lengths on Γ' to satisfy
- genera on Γ' = 0

local RH conditions (filled data in red)

• Start from A 

in Γ Pick leg on Γ' adjacent to v'

in Γ'

$d_v = w(e)$ so $w(e) = 4$ (so $\pi|_{[v,A]} : [0,\infty) \longrightarrow [0,\infty)$
$t \longmapsto 4t$)

• On the other side: $v' \bullet\!\!-\!\!\!-\!\!\bullet \tilde{v}'$ is covered by 2 edges so.

$$w(\tilde{e}_1)\cdot \underset{4}{\underset{\shortparallel}{\ell(\tilde{e}_1)}} = \ell(\tilde{e}') = w(\tilde{e}_2)\, \underset{12}{\underset{\shortparallel}{\ell(\tilde{e}_2)}}$$

so $3\, w(\tilde{e}_2) = w(\tilde{e}_1)$

Also degree of v can be computed from $\tilde{e}'$, so

$$4 = d_v = w(\tilde{e}_1) + w(\tilde{e}_2) = 4\, w(\tilde{e}_2) \quad \text{so} \quad \boxed{\begin{array}{l} w(\tilde{e}_2) = 1 \\ w(\tilde{e}_1) = 3 \end{array}}$$

& $\ell(\tilde{e}') = 1\cdot 4 = 4$.

<u>Now</u>: check RH conditions on v as the 3 ends are added so that RH is satisfied & $d_v = 4$

$$2-2g(v) \overset{?}{=} d_v\,(2-2g(v')) - \sum_{\substack{e\in E(\Gamma)\\ v\in e}} (w(e)-1)$$

$$2-0 \overset{?}{=} 4\,(2-0) - (1-1+3-1+4-1+\underbrace{2-1}) = 8-5-\underbrace{1} = 2.$$

• Move to vertex B, so $d_B = 4$ defines weight 4 on edge of length 7

• We continue in this way until we reach the vertices C & D.

$d_D = 2$ from edge $D \overset{1}{\underset{2}{\bullet\!\!-\!\!\!-\!\!\bullet}} 5$ ⇒ weight on the other side = 2 as well!

Now: $\ell(\pi(D \bullet\!\!-\!\!\bullet E)) = 2\cdot \ell(D \underset{1}{\bullet\!\!-\!\!\bullet} E) = 2\cdot 1 = \boxed{2}$

so $w(E\!-\!C)\; \ell(E \overset{2}{\bullet\!\!-\!\!\bullet} C) = 2$ ⇒ weight = 1

$d_C = 2$ $w(F\!-\!C) + 1 = w(C \overset{}{\underset{1}{\bullet\!\!-\!\!\!-}}) = 2$ so $w(F\!-\!C) = 1$

⇒ get weights 1 on $F\!-\!\infty$ & 3 on $E\!-\!\infty$ by $d_F = 1$, $d_E = 3 = 2+1$. □

<u>Def</u>: • A tropical curve is <u>hyperelliptic</u> if it has degree 2

Example: A degree 4 tropical cover of a genus 0 metric graph Γ' by a genus 5 metric graph Γ. The weights of each edge of Γ with respect to the map $\pi : \Gamma \longrightarrow \Gamma'$ are indicated within $\bigcirc$. The metric graph Γ has a single vertex with positive genus, namely $g(v) = 1$.

The multiple ends are added so that the Riemann–Hurwitz conditions are satisfied for all vertices in Γ

They are marked with arrows