Lecture XXXVIII: Tropical geometry of genus 2 curves II

§1 Tropical hyperelliptic covers:

Recall: Algebraic genus 2 curves are 2-1 covers of $\mathbb{P}^1$ branched at 6 points $(1:\alpha_1), \dots, (1:\alpha_6)$. $\leftrightarrow$ point in $\mathcal{M}_{0,6}$.

Tropically: abstract genus 2 curve Γ is a metric graph & we can realize it as a tropical hyperelliptic curve of a metric tree T with 6 ends.

Tropical hyperelliptic = degree 2 cover with piecewise affine linear behavior on each edge / leg. Weight of edges/legs $\in \mathbb{Z}_{>0}$. $\boxed{[RH]: 2-2g(v) = d_v(2) - \#\{e \ni v, w(e)=2\}}$

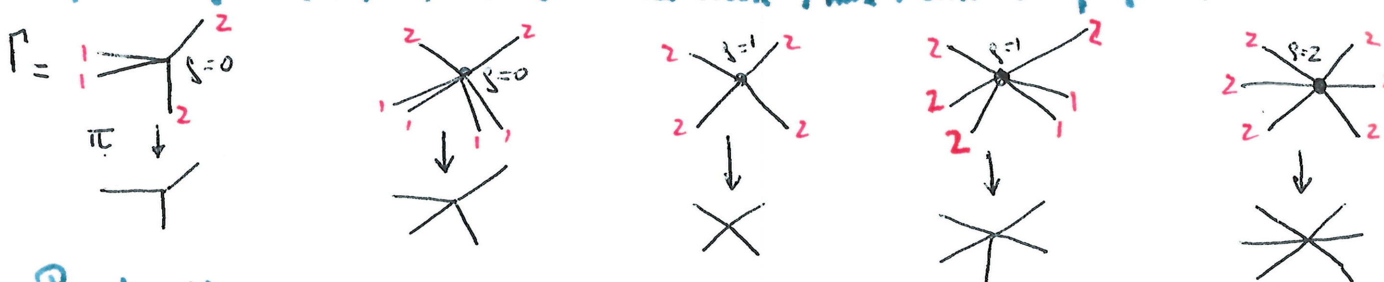
Branch points $\leftrightarrow$ leg of the tree covered by a leg on Γ of weight $= 2$.

Why? $\mathcal{M}_{0,6}^{Trop} = \mathrm{Trop}\, Gr_0(2,6) / \mathbb{R}^6$ via $(K^\times)^6 \setminus \Delta \longrightarrow Gr_0(2,6) \hookrightarrow (K^\times)^{\binom{6}{2}}$

$$\underline{\alpha} = (\alpha_1, \dots, \alpha_6) \longmapsto \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_6 \end{bmatrix} \longmapsto (\alpha_i - \alpha_j)$$

Q: How to find the graph Γ of genus 2 covering each tree? Local description!

Lemma: There are precisely 5 tropical hyperelliptic covers of a single genus 0 vertex of valency $\in \{3,4,5,6\}$ whose source curve Γ has one vertex v of genus ≤ 2.

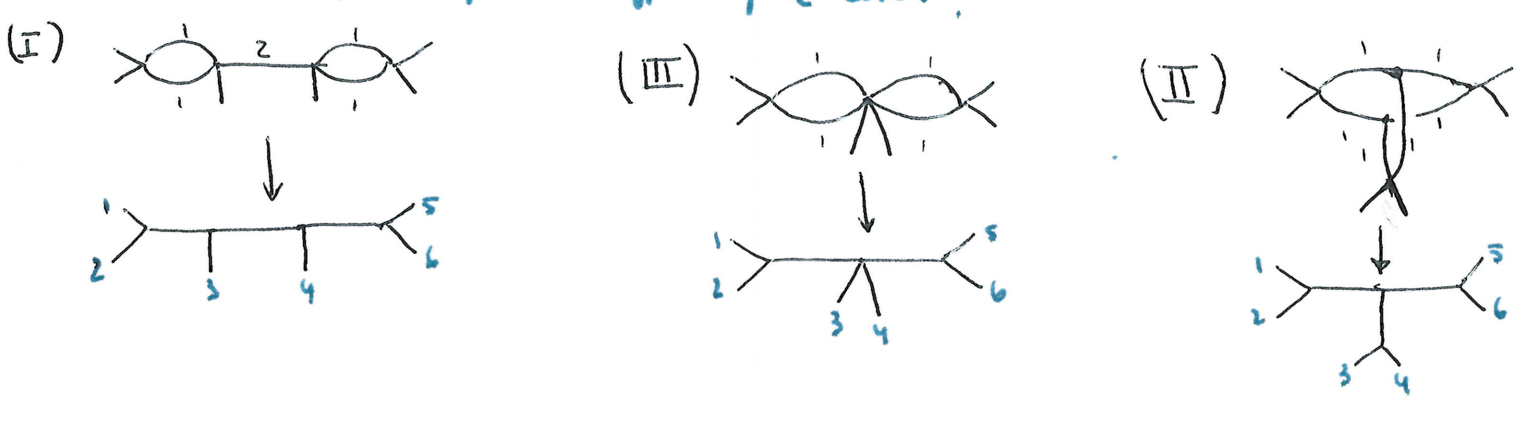


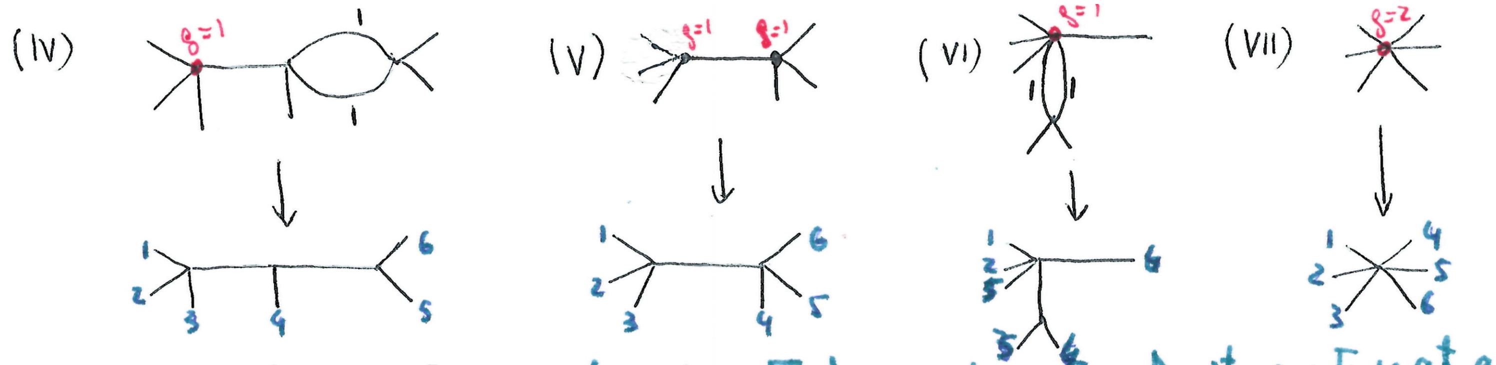
Proof Use RH: $2-2g(v) = \underbrace{2(2-0)}_{=4} - \#\{e \ni v : w(e)=2\}$ $\Rightarrow$ at least one edge of weight 2.

The cover has degree 2 so $w(e) = \begin{cases} 1 & \text{2 edges covering } \pi(e) \\ 2 & \text{only 1 edge covering } \pi(e) \end{cases}$.

List the possible genera $g(v)$ for each valency of $\pi(v)$. $\square$

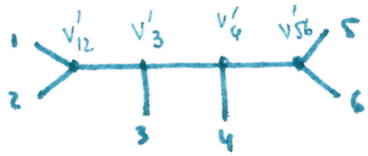
Prop [=M] Each tree on 6 leaves has exactly one genus 2 graph with 6 ends covering it by a tropical hyperelliptic cover.





Proof: Use local description around each vertex from the Lemma and combine them. Do it one type at a time.
All legs are covered by a simple weight 2 edge

(Type I) Tree is caterpillar



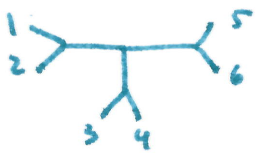
Each vertex is adjacent to a branch point ⇒ only one vertex above each

Start from v'_{3} & v_{3} = only vertex above it

v'_3 connected to v'_{12} & v_3 connected to v_{12}. All vertices are covered by

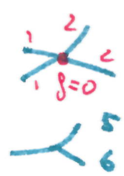
Only one option to glue keeping the degree of the cover = 2.

(Type II) Tree is a snowflake



All local covers are trivalent & weight 1 pairs glue with neighbors ⇒ Theta graph!

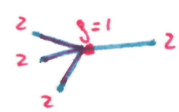
(Type III) One valency 4 vertex in the tree so 2 possible covers:



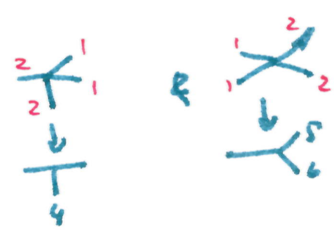
⤳ this is the only one we can glue to

(Type IV)

can only be covered by

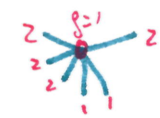


glue to



(Type V) Only one possible cover of each vertex.

(Type VI) Only 1 possible cover for the valency 5 vertex



& glue to unique cover

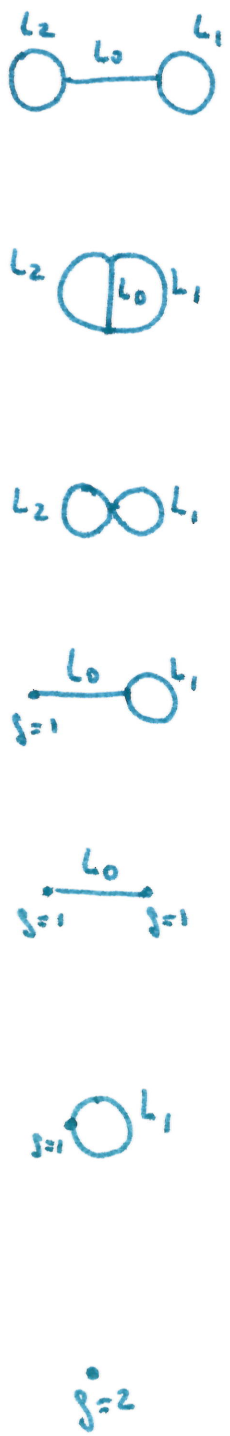
of

(Type VII) Only 1 possible cover.

<u>Next step</u>: Compute the metrics on the trees to get the metrics on the cover

Why? If $e \in E(\Gamma)$ & $\pi(e) = e'$, then $l(e) = \begin{cases} l(e') & \text{if } w(e)=1 \\ l(e')/2 & \text{if } w(e)=2 \end{cases}$

We only need the non-leaf edges of each tree. Use the table with 7 cases (references for each type)



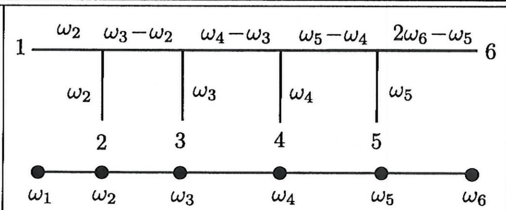
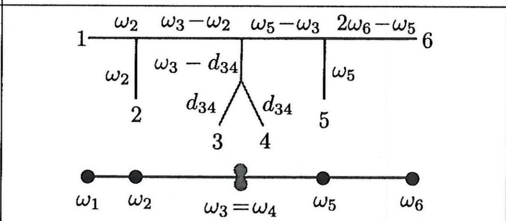
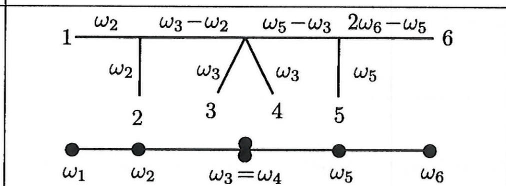
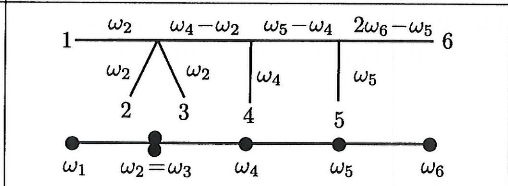
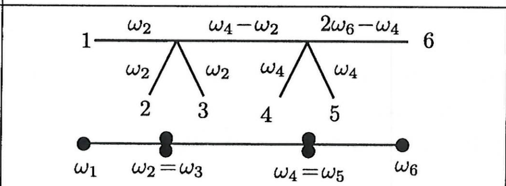
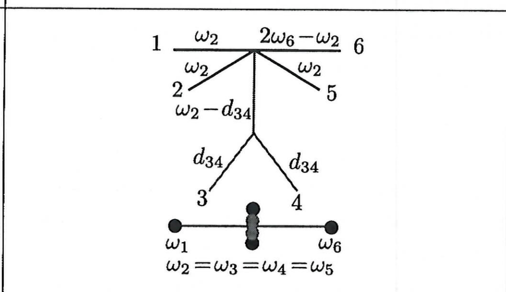
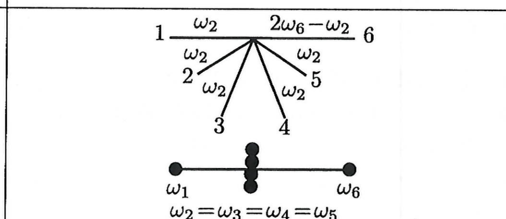
Type	Cover with lengths on $M_{0,6}^{\mathrm{trop}}$	Defining conditions	Lengths on M_2^{trop}
(I)		$\omega_1 < \omega_2 < \omega_3 < \omega_4 < \omega_5 < \omega_6$	$L_0 = (\omega_4 - \omega_3)/2$ $L_1 = 2(\omega_5 - \omega_4)$ $L_2 = 2(\omega_3 - \omega_2)$
(II)		$\omega_1 < \omega_2 < \omega_3 < \omega_5 < \omega_6$ $\omega_3 = \omega_4$ $\mathrm{in}(\alpha_3) = \mathrm{in}(\alpha_4)$	$L_0 = 2(\omega_3 - d_{34})$ $L_1 = 2(\omega_5 - \omega_3)$ $L_2 = 2(\omega_3 - \omega_2)$
(III)		$\omega_1 < \omega_2 < \omega_4 < \omega_5 < \omega_6$ $\omega_3 = \omega_4$ $\mathrm{in}(\alpha_3) \neq \mathrm{in}(\alpha_4)$	$L_0 = 0$ $L_1 = 2(\omega_5 - \omega_3)$ $L_2 = 2(\omega_3 - \omega_2)$
(IV)		$\omega_1 < \omega_2 < \omega_4 < \omega_5 < \omega_6$ $\omega_2 = \omega_3$ $\mathrm{in}(\alpha_2) \neq \mathrm{in}(\alpha_3)$	$L_0 = (\omega_4 - \omega_2)/2$ $L_1 = 2(\omega_5 - \omega_4)$ $L_2 = 0$
(V)		$\omega_1 < \omega_2 < \omega_4 < \omega_6$ $\omega_2 = \omega_3$, $\omega_4 = \omega_5$ $\mathrm{in}(\alpha_2) \neq \mathrm{in}(\alpha_3)$ $\mathrm{in}(\alpha_4) \neq \mathrm{in}(\alpha_5)$	$L_0 = (\omega_4 - \omega_2)/2$ $L_1 = 0$ $L_2 = 0$
(VI)		$\omega_1 < \omega_2 < \omega_6$ $\omega_2 = \omega_3 = \omega_4 = \omega_5$ $\mathrm{in}(\alpha_2) \neq \mathrm{in}(\alpha_5)$ $\mathrm{in}(\alpha_2), \mathrm{in}(\alpha_5) \neq \mathrm{in}(\alpha_3)$ $\mathrm{in}(\alpha_3) = \mathrm{in}(\alpha_4)$	$L_1 = 2(\omega_2 - d_{34})$ $L_1 = 0$ $L_2 = 0$
(VII)		$\omega_1 < \omega_2 < \omega_6$ $\omega_2 = \omega_3 = \omega_4 = \omega_5$ $\mathrm{in}(\alpha_i) \neq \mathrm{in}(\alpha_j)$ for $1 < i < j < 6$	$L_0 = 0$ $L_1 = 0$ $L_2 = 0$

TABLE 1. Combinatorial types with the corresponding defining valuation conditions, and length data for $M_{0,6}^{\mathrm{trop}}$ and M_2^{trop}. Here, $\omega_i = -\mathrm{val}(\alpha_i)$, $d_{34} = -\mathrm{val}(\alpha_3 - \alpha_4)$, and the edge lengths L_0, L_1 and L_2 refer to Figure 3.

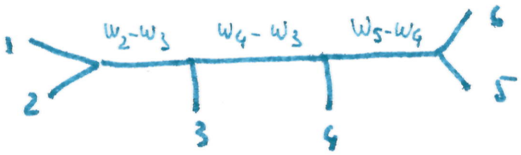
Use the Plücker embedding & the tropical values: $d_{ij} = -\mathrm{val}(\alpha_i - \alpha_j)$ $1 \le i < j \le 6$.

WLOG, we ~~write~~ $w_i = -\mathrm{val}(\alpha_i)$ & assume $w_1 \le w_2 \le w_3 \le w_4 \le w_5 \le w_6$.
($\mathrm{val}\,\alpha_1 \geq \mathrm{val}\,\alpha_2 \geq \cdots \geq \mathrm{val}\,\alpha_6$.)

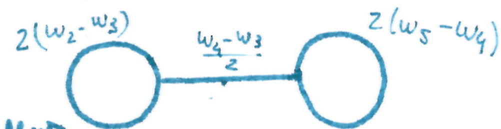
- Type (I): Tropical Plücker vector from $w_1 < w_2 < w_3 < w_4 < w_5 < w_6$ becomes:

$$-\mathrm{val}(P_\bullet) = (\overset{12}{w_2}, \overset{13}{w_3}, \overset{14}{w_4}, \overset{15}{w_5}, \overset{16}{w_6}, \overset{23}{w_3}, \overset{24}{w_4}, \overset{25}{w_5}, \overset{26}{w_6}, \overset{34}{w_4}, \overset{35}{w_5}, \overset{36}{w_6}, \overset{45}{w_5}, \overset{46}{w_6}, \overset{56}{w_6})$$

The 4-Pt condition tells us what the tree looks like:



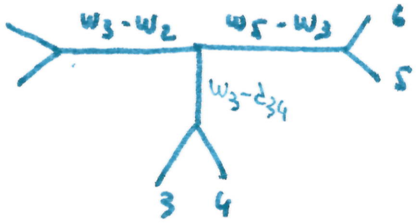
$\rightsquigarrow$ trop hyperelliptic cover



- Type (II): We have $w_1 \le w_2 < w_3 = w_4 < w_5 < w_6$ & $\underbrace{\mathrm{val}(\alpha_3 - \alpha_4)}_{=:-d_{34}} > \mathrm{val}(\alpha_3)$

$$-\mathrm{val}(P) = (w_2, w_3, w_4, w_5, w_6, w_3, w_4, w_5, \overset{26}{w_6}, \overset{34}{d_{34}}, w_5, w_6, w_5, w_6, w_6)$$

The 4 Pt condition tells us what the tree looks like:



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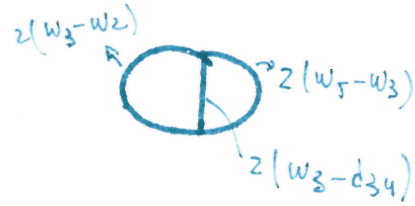

- Lower dimensional types = specialization of Type (I) and/or (II), so we can specialize edge length formulas. (See Table)

Remark: Any choice $(\alpha_1, \ldots, \alpha_6) \in (K^*)^6$ can be turned into one of the 7 cases in the table by an automorphism of $\mathbb{P}^1$. So we get the minimal skeleton of C^{an} from the valuations of α_i & $\alpha_i - \alpha_j$!

§2 Igusa invariants & their tropicalization:

Isomorphism classes of genus 2 curves X/K for char $K \neq 2$, given by $y^2 = u \prod_{i=1}^{6} (x - \alpha_i)$ with $\alpha_i \neq \alpha_j$ $\forall i \neq j$ are characterized by 3 Igusa invariants j_1, j_2, j_3. They are degree 0 rational functions in $(\alpha_1, \ldots, \alpha_6)$ with $\mathbb{Z}$-coefficients

- Mestre: Given $j_1, j_2, j_3 \in K$, build X/L with $j_i(X) = j_i$ for $L|K$ field extension.

Write $\Delta_{ij} = (\alpha_i - \alpha_j)^2$ $1 \le i \neq j \le 6$. $i \neq j$

Def $j_1(X) = \dfrac{A^5}{D}$, $j_2(X) = \dfrac{A^3 B}{D}$, $j_3(X) = \dfrac{A^2 C}{D}$ where A, B, C, D are

$$A = u^2 \sum_{(ij),(k\ell),(m,n)} \Delta_{ij}\Delta_{k\ell}\Delta_{mn}$$ (15 tripartitions of $[6]=\{1,\ldots,6\}$)

$$B := u^4 \sum_{ijk,\,\ell mn} \Delta_{ij}\Delta_{jk}\Delta_{ki}\,\Delta_{\ell m}\Delta_{mn}\Delta_{n\ell}$$ (10 partitions of $[6]$ of type $(3,3)$)

$$C = u^6 \sum_{\substack{(ijk,\,\ell mn)\\(i\ell,\,jm,\,kn)}} \Delta_{ij}\Delta_{jk}\Delta_{ki}\,\Delta_{\ell m}\Delta_{mn}\Delta_{n\ell}\,\Delta_{i\ell}\Delta_{jm}\Delta_{kn}$$ (60 summands)

$$D = \text{Discriminant}^2 = u^{10}\prod_{1\leq i<j\leq 6}\Delta_{ij}.$$

Def $j_i^{trop}(\text{Trop}\,X) := -\text{val}(j_i(X))$ $\quad i=1,2,3$ Tropical I genus invariants (independent of iso class of X)

Thm [- Markwig] $j_1^{trop}, j_2^{trop}, j_3^{trop}$ are continuous piecewise linear functions on Π_2^{trop}, linear on each cell.

(I) $j_1{}^{trop}_{\text{o-o}}(\text{Trop}\,X) = L_1 + 12L_0 + L_2$, $\quad j_2{}^{trop}_{\text{o-o}}(\text{Trop}\,X) = j_3{}^{trop}_{\text{o-o}}(\text{Trop}\,X) = L_1 + 8L_0 + L_2$

(II) $j_1{}^{trop}_{\text{①}}(\text{Trop}\,X) = j_2{}^{trop}_{\text{①}}(\text{Trop}\,X) = j_3{}^{trop}_{\text{①}}(\text{Trop}\,X) = L_1 + L_0 + L_2$.

The invariant $j_4 = j_2 - 4j_3$ & its tropicalization satisfies the same continuous PL behavior on Π_2^{trop}, with $j_4{}^{trop}_{\text{o-o}} = j_2{}^{trop}_{\text{o-o}}$ & $j_4{}^{trop}_{\text{①}} = L_1 + L_0 + L_2 - \min\{L_0, L_1, L_2\}$

All formulas are valid under specialization.

Proof: We use the canonical 7 regions in the table.

• Type (I) = o-o $\quad$ Since $w_1 < w_2 < w_3 < w_4 < w_5 < w_6$, we get

$-\text{val}(A) = 2(w_4 + w_5 + w_6)$

$-\text{val}(B) = 4(w_5 + w_6) + 2(w_3 + w_4)$

$-\text{val}(C) = 6(w_5 + w_6) + 4w_4 + 2w_3$

$-\text{val}(D) = 2w_2 + 4w_3 + 6w_4 + 8w_5 + 10w_6$

[Use GB with w-weight]. $w_i = -\text{val}(\alpha_i)$]

Then $j_1^{trop} = -5\,\text{val}(A) + \text{val}(D) \underset{\substack{\downarrow\\ \text{formulas for}\\ L_0, L_1, L_2}}{=} L_1 + 12L_0 + L_2$

$$j_2^{trop} = \underbrace{-3\,\text{val}(A) - \text{val}(B)}_{= -2\,\text{val}(A) - \text{val}(C)} + \text{val}(D) = L_1 + 8L_0 + L_2 = j_3^{trop}.$$

• Type II: Replace Δ_{34} by a new variable α_{34}^2 & replace α_4 by $\alpha_3 + \alpha_{34} \rightsquigarrow A', B', C', D'$

weight of $\alpha_{34} = d_{34}$. Explicit GB computation gives

$-\text{val}(A') = 2(w_3 + w_5 + w_6)$, $\quad -\text{val}(B') = 4(w_3 + w_5 + w_6)$, $\quad -\text{val}(C') = 6(w_3 + w_5 + w_6)$

$-\text{val}(D') = 2w_2 + 8w_3 + 2d_{34} + 8w_5 + 10w_6$

- Specialization w-initial terms on lower dimensional cells of each A, B, C, D contain terms for $\tilde{w}$-initial forms on top-dimensional cells for $\tilde{w} \rightsquigarrow w$. So formulas remain valid on Types (III) – (VII).

Q: Why the shape of j_4?

Use linear relation: $\mathrm{val}(A) + \mathrm{val}(B) = \mathrm{val}(C)$. $\rightsquigarrow$ $AB - \lambda C$ may improve things!

Type I: $\mathrm{in}_w A \, \mathrm{in}_w B = 24\, \alpha_3^2 \alpha_4^2 \alpha_5^6 \alpha_6^6 = 3\, \mathrm{in}_w C \rightsquigarrow \lambda = 3$

Type II: $\mathrm{in}_w A \, \mathrm{in}_w B = 32\, \alpha_3^6 \alpha_5^6 \alpha_6^6 = 4\, \mathrm{in}_w C \rightsquigarrow \lambda = 4$

- Try $j_4' = \dfrac{A^2(AB - 3C)}{D} = j_2 - 3 j_3$ but $j_4'^{\,\mathrm{trop}}{}_{0\text{-}0} = L_1 + 6L_0 + L_2$ so we don't get new length data

- Try $j_4 = \dfrac{A^2(AB - 4C)}{D} = j_2 - 4 j_3$. Same GB-type computations give

$$-\mathrm{val}(AB - 4C) = \begin{cases} -\mathrm{val}(A) - \mathrm{val}(B) & \text{for } w \text{ giving Type (I)} \\ -\min\{L_0, L_1, L_2\} & \text{for } w \text{ giving Type (II)}. \end{cases}$$

$\square$