MATH 8140 - Topics in Algebraic Geometry (Riemann surfaces)

Homework 1

Basic definitions of Riemann surfaces and holomorphic maps

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams. If you do so, please indicate the name of your collaborator(s) on your submission.

Solutions to each problem can be uploaded (on Carmen) by at most one student. There is no deadline, so work at your own pace.

Please use the following name for the file you upload `HW#_Problem#.pdf`.

Problem 1. (Riemann surfaces from graphs of holomorphic functions)
Fix a connected open subset V of $\mathbb{C}$.

(i) Let $g \in \mathcal{O}(V)$ be a holomorphic function on V. Consider the graph X of g inside $\mathbb{C}^2$, i.e.

$$X = \{(z, g(z)) | z \in V\}.$$

Show that $(X, \phi : X \to V)$ determines a Riemann structure on X.

(ii) Generalize this construction to the case of multiple holomorphic functions $g_1, \ldots, g_s \in \mathcal{O}(V)$.

Recall: given two $\mathbb{R}$-linearly independent vectors ω_1 and ω_2 in $\mathbb{C}$, consider the rank-2 discrete lattice $\Gamma = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ and the complex tori $\mathbb{E} = \mathbb{C}/\Gamma$ constructed in Lecture 1 (called Elliptic curves)

Problems 2, 3 and 4 involve discrete lattices on $\mathbb{C}$ and their associated complex tori.

Problem 2. Show that the map $\psi \colon \mathbb{C} = \mathbb{R}\omega_1 \oplus \mathbb{R}\omega_2 \to \mathbb{S}^1 \times \mathbb{S}^1$ defined by

$$\psi(\lambda_1\omega_1 + \lambda_2\omega_2) = (e^{2\pi\iota\lambda_1}, e^{2\pi\iota\lambda_2})$$

factors through a homeomorphism $\mathbb{E} \to \mathbb{S}^1 \times \mathbb{S}^1$.

Problem 3 (Forster §1.4)**.** Let $\Gamma = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ and $\Gamma' = \mathbb{Z}\omega_1' \oplus \mathbb{Z}\omega_2'$ be two lattices in $\mathbb{C}$ show that $\Gamma = \Gamma'$ if, and only if, there exists a matrix $A \in \mathrm{SL}(2, \mathbb{Z}) = \{A \in \mathrm{GL}(2, \mathbb{Z}) : \det(A) = 1\}$ such that $A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}$.

Problem 4 (Forster §1.5)**.** This problem discusses how to pick lattices giving isomorphic complex structures on tori.

(i) Let Γ, Γ' be two discrete rank-2 lattices in $\mathbb{C}$ as constructed in Problem 3. Assume there exists $\alpha \in \mathbb{C}^*$ with $\alpha\Gamma \subseteq \Gamma'$. Show that the map $\mathbb{C} \to \mathbb{C}$ with $z \mapsto \alpha z$ induces a holomorphic map $\mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$, which is biholomorphic if, and only if, $\alpha\Gamma = \Gamma'$.

(ii) Show that every torus $X = \mathbb{C}/\Gamma$ is isomorphic to a torus of the form

$$X_\tau := \mathbb{C}/(\mathbb{Z}1 \oplus \mathbb{Z}\tau), \tag{1}$$

where $\tau \in \mathbb{C}$ satisfies $\mathrm{Im}(\tau) > 0$.

(iii) Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ and fix τ in $\mathbb{C}$ with $\mathrm{Im}(\tau) > 0$. Let

$$\tau' := \frac{a\,\tau + b}{c\,\tau + d}.$$

Show that the tori $X(\tau)$ and $X(\tau')$ (see (1)) are isomorphic Riemann surfaces.

Problem 5 (Forster §1.2)**. (Moebius transformations on $\mathbb{C}$)**
Fix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$ and consider the rational map

$$f\colon \mathbb{C} \dashrightarrow \mathbb{C} \qquad f(z) = \frac{a\,z + b}{c\,z + d}.$$

(i) Show that f can be extended to a meromorphic function $f\colon \mathbb{P}^1 \dashrightarrow \mathbb{C}$ away from $\{z \in \mathbb{C} : c\,z + d \neq 0\}$.

(ii) Show that f can be extended to a holomorphic function $f\colon \mathbb{P}^1 \to \mathbb{P}^1$.

(iii) Show that $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ is biholomorphic.

Recall from the local behavior of holomorphic functions $f\colon X_1 \to X_2$ between Riemann surfaces, that given any point $a_1 \in X_1$ and a neighborhood $U_0 \subset X_1$ of a_1, we can find a neighborhood $W \subset X_2$ of $a_2 := f(a_1)$ and an open U with $a_1 \in U \subset U_0$ such that $f^{-1}y \cap U$ has the same size for each $y \in W \smallsetminus \{a_2\}$. This number is called the **ramification index** of f at a_1 (or the multiplicity of f at a_1).

Problems 6, 7 an 8 below involve ramification indices.

Problem 6. Fix a degree N polynomial F in $\mathbb{C}[z]$.

(i) Show that F defines a holomorphic function $f\colon \mathbb{P}^1 \to \mathbb{P}^1$, with $f(z) = F(z)$ for each $z \neq \infty$ and $f(\infty) = \infty$.

(ii) Show that the ramification index of f is locally constant.

(iii) Compute the ramification index of f at each point of $\mathbb{P}^1$. (*Hint:* Compute this index at ∞.)

Problem 7. Show that the ring of meromorphic functions $\mathcal{M}(\mathbb{P}^1)$ is the field of rational functions $\mathbb{C}(z)$ in one variable, i.e. every meromorphic function on $\mathbb{P}^1$ is a quotient of polynomials in z.
(*Hint:* Show that for each $f \in \mathcal{M}(\mathbb{P}^1)$ the fiber $f^{-1}(\infty)$ is discrete and finite.)

Problem 8. If $f\colon X_1 \to X_2$ is an injective holomorphic function between Riemann surfaces, show that f is biholomorphic onto its image. (*Hint:* Determine the ramification index of f.)

Problem 9. (Stereographic projection)
Consider the north pole $N = (0,0,1)$ in the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ and let $P = (0,0,-1)$ be the south pole. Fix a complex atlas on $\mathbb{S}^2$ via the stereographic projections from the south and north poles, respectively. More precisely, consider the covering $\{U_0, U_1\}$ of $\mathbb{S}^2$ and the maps ϕ_0 and ϕ_1 with:

$$\phi_0\colon U_0 = \mathbb{S}^2 \smallsetminus \{P\} \to \mathbb{C} \quad \text{ and } \quad \phi_1\colon U_1 = \mathbb{S}^2 \smallsetminus \{N\} \to \mathbb{C},$$

defined by $\phi_0(x,y,z) = \frac{x - \iota\, y}{1+z}$ and $\phi_1(x,y,z) = \frac{x + \iota\, y}{1-z}$.

(i) Show that this defines a complex structure on the unit sphere. We call this the Riemann sphere.

(ii) Show that this Riemann sphere is isomorphic to $\mathbb{P}^1$ by showing ϕ_1 extends to a unique biholomophism $\rho\colon \mathbb{S}^2 \to \mathbb{P}^1$ with $\rho(N) = \infty$.