MATH 8140 - Topics in Algebraic Geometry (Riemann surfaces)

Homework 2

Fundamental groups, Coverings, path lifting properties, proper maps

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams. If you do so, please indicate the name of your collaborator(s) on your submission.

Solutions to each problem can be uploaded (on Carmen) by at most one student. There is no deadline, so work at your own pace.

Please use the following name for the file you upload `HW#_Problem#.pdf`.

Problem 1 (Forster §3.2)**.** Suppose that X and Y are two pathwise connected topological spaces. Prove that $\pi_1(X \times Y) \simeq \pi_1(X) \times \pi_1(Y)$.

Definition: Suppose X and Y are two topological spaces, and pick $a \in X$ and $b \in Y$. Let $f, g\colon X \to Y$ be two continuous functions with $f(a) = g(a) = b$. We say f and g are *homotopic* if there exists a continuous map $F\colon X \times [0,1] \to Y$ such that

1. $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$, and
2. $F(a, t) = b$ for all $t \in [0, 1]$

Problem 2 (Forster §3.3)**.** Let X and Y be two topological spaces with distinguished points $a \in X$ and $b \in Y$. and $f, g\colon X \to Y$ to continuous functions with $f(a) = g(a) = b$. Consider the induced maps

$$f_*, g_*\colon \pi_1(X, a) \to \pi_1(Y, b).$$

Show that $f_* = g_*$ if f and g are homotopic.

Problems 3 and 4 below involve the following two Hausdorff and connected topological sets:

$$X := \mathbb{C} \smallsetminus \{\frac{2k+1}{2}, k \in \mathbb{Z}\} \quad \text{and} \quad Y := \mathbb{C} \smallsetminus \{\pm 1\}. \tag{1}$$

Problem 3 (Forster §4.1)**.** Show that $\sin : X \to Y$ is a covering map where X and Y are the sets from (1).

Problem 4 (Forster §4.1)**. (A non-abelian fundamental group)**
Consider the set Y from (1) and the following two closed curves on X:

$$u, v\colon [0,1] \to Y \quad u(t) = 1 - e^{2\pi \iota t}, \quad v(t) = -1 + e^{2\pi \iota t} \ \text{ for } t \in [0,1].$$

Let $\hat{u}, \hat{v}\colon [0,1] \to X$ be the unique lifts of u, v relative to $\sin\colon X \to Y$ with $\hat{u}(0) = \hat{v}(0) = 0$.

(i) Show that $\hat{u}(1) = 2\pi$ and $\hat{v}(1) = -2\pi$.

(ii) Conclude from the previous item that $\pi(Y)$ is non-abelian.

Problem 5 (Forster §4.2)**.** Fix two pathwise connected Hausdorff topological spaces X, Y and a covering map $f\colon X \to Y$. Show that the induced map $f_*\colon \pi_1(X) \to \pi_1(Y)$ is injective.

Problem 6 (Forster §4.3)**. (Lifting criteria via fundamental groups)**

Let X and Y be two Hausdorff topological spaces and let $p\colon X \to Y$ be a covering map. Fix a topological space Z and a continuous map $f\colon Z \to Y$. Pick $c \in Z$ and $a \in X$ with $f(c) = p(a)$. Assume that Z is connected and locally pathwise connected.

Prove that there exists a lifting $\hat{f}\colon Z \to X$ of f relative to p with $f(c) = a$ if, and only if, $f_*(\pi(Z, c)) \subseteq p_*(\pi(Y, f(a))$.

Problem 7. Consider a Riemann surface X and let F be a discrete and close subset of X. Show that $X' = X \smallsetminus F$ is also a Riemann surface. (*Hint:* Show that X' is pathwise connected.)

Problem 8 (Forster §4.5)**.** Determine the ramification points of the map $f\colon \mathbb{C} \to \mathbb{P}^1$ with $f(z) = \frac{1}{2}\,(z+1/z)$.

Problem 9. Consider the map $p\colon \mathbb{C} \to \mathbb{C}$ defined by $p(z) = z^3 - 3z$.

(i) Show that 1 and -1 are the only branch points of p.

(ii) Determine the critical values and their fibers.

(iii) Prove that restriction $p\colon \mathbb{C} \smallsetminus \{\pm 1, \pm 2\} \to \mathbb{C} \smallsetminus \{\pm 2\}$ is a 3-sheeted holomorphic covering map.