# MATH 8140 - Topics in Algebraic Geometry (Riemann surfaces)

## Homework 3

## Coverings, Galois correspondence, sheaves and analytic continuation

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams. If you do so, please indicate the name of your collaborator(s) on your submission.

Solutions to each problem can be uploaded (on Carmen) by at most one student. There is no deadline, so work at your own pace.

Please use the following name for the file you upload `HW#_Problem#.pdf`.

**Problem 1.** Fix two manifolds $X, Y$ and a covering $q\colon X \to Y$. Let $p\colon \tilde{Y} \to Y$ be the universal cover of $Y$ and pick $F\colon \tilde{Y} \to X$ with $q \circ F = p$. Define $G := \operatorname{Deck}(\tilde{Y}|X)$ to be the group of Deck transformations associated to the map $F$.

(i) Show that $q$ is a Galois covering if and only if $G$ is normal in $\operatorname{Deck}(\tilde{Y}|Y)$.

(ii) Assuming $G$ is normal in $\operatorname{Deck}(\tilde{Y}|Y)$, show that $\operatorname{Deck}(X|Y) \simeq \operatorname{Deck}(\tilde{Y}|Y)/G$.

**Problem 2** (Forster §5.4)**.** Fix two rank-two lattices $\Gamma, \Gamma' \subset \mathbb{C}$ and a non-constant holomorphic map

$$f\colon \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$$

with $f(0) = 0$.

(i) Show that there exists a unique $\alpha \in \mathbb{C}^*$ such that $\alpha\Gamma \subset \Gamma'$ making the following diagram commute:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathbb{C} \arrow[d, "\pi'"] \\ \mathbb{C}/\Gamma \arrow[r, "f"] & \mathbb{C}/\Gamma' \end{tikzcd}$$

where $F(z) = \alpha\, z$ and $\pi, \pi'$ are the canonical projections.

(ii) Prove that $f$ is an unbranched covering map and $\operatorname{Deck}(\mathbb{C}/\Gamma \xrightarrow{f} \mathbb{C}/\Gamma') \simeq \Gamma'/\alpha\Gamma$.

**Problem 3** (Forster §5.5)**.** Fix $X = \mathbb{C}\smallsetminus\{\pm1, \pm2\}$, $Y = \mathbb{C}\smallsetminus\{\pm2\}$, an let $p\colon X \to Y$ be the map $p(z) = z^3 - 3z$. By Problem 9 in HW 2, we know that $p$ is a proper unbranched 3-sheeted holomorphic map (i.e. a covering with 3 sheets). Compute $\operatorname{Deck}(X|Y)$ and show that the covering $p$ is not Galois.

**Problem 4** (Forster §5.7)**.** Suppose $X$ and $Y$ are connected Hausdorff spaces. Show that any 2-sheeted covering map $p\colon X \to Y$ is Galois.

**Problem 5** (Forster §7.1)**.** Suppose that $X$ and $Y$ are Riemann surfaces, $p\colon X \to Y$ is a proper holomorphic covering (i.e., without branch points) and let $f\colon X \to \mathbb{C}$ be a holomorphic map. Fix $a \in X$ and consider the germ $\varphi = p_*(\rho_a(f)) \in \mathcal{O}_b$ for $b := p(a) \in Y$.

Prove that $(X, p, f, b)$ is a maximal analytic continuation of $\varphi$ if and only if the following condition is satisfied: "For any two distinct points $a_1, a_2 \in p^{-1}(b)$, the germs $\varphi_1 = p_*(\rho_{a_1}(f))$ and $\varphi_3 = p_*(\rho_{a_2}(f))$ in $\mathcal{O}_b$ are different."

*Definition:* Consider a topological space $X$ and a presheaf $\mathcal{F}$ on $X$. Given an open $U \subset X$, a *section to* the surjective map $p : |\mathcal{F}| \to X$ over $U$ is given by a map $s\colon U \to |\mathcal{F}|$ with $p \circ s = \mathrm{inc}_U : U \hookrightarrow X$.

Typically, we refer to elements $s \in \mathcal{F}(U)$ as "sections" of $\mathcal{F}$ on $U$. The next problem justifies the use of this terminology.

**Problem 6 ("Section" terminology for sheaves).** Fix $X$ a topological space and a presheaf of sets $\mathcal{F}$ on $X$. For each open $U \subset X$, consider the set

$$\mathcal{G}(U) := \{s\colon U \to |\mathcal{F}| \text{ continuous sections to } p \text{ over } U\}$$

Consider the map $\phi\colon \mathcal{F}(U) \to \mathcal{G}(U)$ defined by $\phi(f) = (s\colon x \mapsto \rho_x(f) \text{ for all } x \in U)$.

(i) Show that $\mathcal{G}$ defines a sheaf on $X$ with the usual restriction.

(ii) Show that $\phi$ is well-defined, i.e. $\phi(f) \in \mathcal{G}(U)$ for each $f \in \mathcal{F}(U)$.

(iii) Show that $\phi$ is a map of presheaves and that the induced map on stalks is a bijection.

(iv) Confirm that $\phi$ is a bijection when $\mathcal{F}$ is a sheaf. This justifies the terminology "sections" for elements in $\mathcal{F}(U)$.