MATH 8140 - Topics in Algebraic Geometry (Riemann surfaces)

Homework 4

Algebraic functions, differential forms, integration of 1-forms

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams. If you do so, please indicate the name of your collaborator(s) on your submission.

Solutions to each problem can be uploaded (on Carmen) by at most one student. There is no deadline, so work at your own pace.

Please use the following name for the file you upload `HW#_Problem#.pdf`.

Problem 1 (Forster §8.2)**.** Let X, Y be two compact Riemann surfaces, and fix two collections $A = \{a_1, \ldots, a_n\} \subset X$ and $B := \{b_1, \ldots, b_m\} \subset Y$ of distinct points. Set $X' := X \smallsetminus A$ and $Y' := Y \smallsetminus B$. Show that every biholomorphism $f\colon X' \to Y'$ extens to a biholomorphism $f\colon X \to Y$.

Recall that $\mathbb{C}\{\{z\}\}$ denotes the field of convergent power series in a neighborhood of the origin.

Problem 2 (Forster §8.3)**.** Let $F(z, w) := w^2 - z^3 w + z \in \mathbb{C}\{\{z\}\}[w]$.

(i) Show that F is irreducible over $\mathbb{C}\{\{z\}\}[w]$.

(ii) Determine the Puiseux expansion of a solution $w(\eta)$ to $F(\eta^2, w(\eta)) = 0$.

Problem 3. Fix a Riemann surface X and open $U \subset X$ and let $k = 0$ or 1. Show that $F^*(w)$ for $w \in \mathcal{E}^{(k)}(U)$ is independent of the expressions defining w (i.e. $w = \sum_j f_j\, dg_j$ with $f_j, g_j \in \mathcal{E}(U)$ for $k = 1$, and $w = \sum_j f_j\, dg_j \wedge dh_j$ with $f_j, g_j, h_j \in \mathcal{E}(U)$ for $k = 2$, respectively).

Problem 4. Consider the Riemann surface $\mathbb{C}^*$ and the (closed) holomorphic 1-form $w = \frac{dz}{z}$.

(i) Determine the period homomorphism $\rho_w\colon \pi_1(\mathbb{C}^*) \to \mathbb{C}$.

(ii) Compute p^*w where $p = \exp\colon \mathbb{C} \to \mathbb{C}^*$ is the universal covering of $\mathbb{C}^*$.

Definition: Given a Riemann surface X and its universal covering $\pi\colon \tilde{X} \to X$, we say a function $f \in \mathcal{E}(\tilde{X}))$ is *additively automorphic with constant summands of automorphy* if for each $\sigma \in \mathrm{Deck}(\tilde{X}|X)$ the function

$$a_\sigma := f - \sigma \cdot f \in \mathcal{E}(\tilde{X})$$

is constant.

Problem 5. Consider the holomorphic 1-form $w = \frac{dz}{z}$ on $\mathbb{C}^*$ from Problem 4. Fix any primitive F to p^*w on $\mathbb{C}$.

(i) Show that F is an additively automorphic function with constant summands of automorphy.

(ii) Show that F is not constant along the fibers of the universal cover $p = exp\colon \mathbb{C} \to \mathbb{C}^*$.

Problem 6 (Forster §9.2)**.** Consider the holomorphic 1-form $w = dz/(1 + z^2)$ on $\mathbb{C} \smallsetminus \{\pm i\}$.

(i) Show that w can be extended to a holomorphic 1-form on $\mathbb{P}^1 \smallsetminus \{\pm i\}$.

(ii) Determine p^*w, where $p = tan\colon \mathbb{C} \to \mathbb{P}^1 \smallsetminus \{\pm i\}$ is the universal covering.

The next problem shows how to determine the periods of an elliptic curve and highlights the relevance of the construction of the summands of automorphy.

Problem 7. Consider a rank-two lattice $\Gamma \subset \mathbb{C}$ and the associated Riemann surface $X = \mathbb{C}/\Gamma$.

(i) Show that $\mathrm{Deck}(\mathbb{C}|X) \simeq \Gamma$.

(ii) Show that the identity function $f = \mathrm{id}\colon \mathbb{C} \to \mathbb{C}$ is an additively automorphic function with constant summands of automorphy. Furthermore, confirm that $a_\sigma = \sigma$ for each $\sigma \in \Gamma$.

(iii) Build a closed 1-form w on X with $dz = \pi^*(w)$, where $\pi\colon \mathbb{C} \to X$ is the natural projection.

(iv) Determine $\rho_w(\sigma)$ for each $\sigma \in \Gamma$.