MATH 8140 - Topics in Algebraic Geometry (Riemann surfaces)

Homework 5

Čech cohomology

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams. If you do so, please indicate the name of your collaborator(s) on your submission.

Solutions to each problem can be uploaded (on Carmen) by at most one student. There is no deadline, so work at your own pace.

Please use the following name for the file you upload `HW#_Problem#.pdf`.

Problem 1 (Forster §12.1)**.** Fix n distinct points $p_1, \ldots, p_n$ on $\mathbb{C}$ and let $X := \mathbb{C} \smallsetminus \{p_1, \ldots, p_n\}$. Show that $H^1(X, \underline{\mathbb{Z}}) \simeq \mathbb{Z}^n$.
(*Hint:* Find a covering $\mathcal{U} := (U_1, U_2)$ of X by simply connected opens, where $U_1 \cap U_2$ has $n+1$ connected components.)

Recall the notation $V \Subset U$ for a *relatively compact* open subset V of U, meaning that $\overline{V}$ is compact and $\overline{V} \subset U$.

Problem 2 (Forster §12.2 (a))**.** Let X be a differentiable manifold, and let U, V be opens with $V \subset U$ and $V \Subset U$. Show that V meets only a finite number of connected components of U.

Problem 3 (Forster §12.2 (b))**.** Let X be a compact differentiable manifold and pick two open *finite* coverings $\mathcal{U} := (U_i)_{i \in I}$ and $\mathcal{V} := (V_j)_{j \in J}$ such that $V_i \Subset U_i$ for all $i \in I$. Show that the image of the restriction map

$$Z^1(\mathcal{U}, \underline{\mathbb{C}}) \to Z^1(\mathcal{V}, \underline{\mathbb{C}})$$

is finite-dimensional over $\mathbb{C}$.

Problem 4 (Forster §12.2 (c))**.** Let X be a compact Riemann surface. Show that $H^1(X, \mathbb{C})$ is a finite dimensional vector space over $\mathbb{C}$. (*Hint:* Use finite coverings $\mathcal{U}$, $\mathcal{V}$ as is Problem 3, with U_i and V_i homeomorphic to open disks in $\mathbb{C}$.)

Problem 5 (Forster §12.3 (a))**.** Let X be a compact Riemann surfaces, and consider the sheaves of locally constant functions $\underline{\mathbb{Z}} \subset \underline{\mathbb{C}}$.

1. Show that this inclusion of sheaves gives an associated map on cohomology:
$$\rho \colon H^1(X, \underline{\mathbb{Z}}) \to H^1(Z, \underline{\mathbb{C}})$$
2. Show that ρ is injective.

Problem 6 (Forster §12.3 (b))**.** Let X be a compact Riemann surface.

1. Show that $H^1(X, \underline{\mathbb{Z}})$ is a finitely generated abelian group using the technique from Problem 4.
2. Use Problem 5 to conclude that $H^1(X, \underline{\mathbb{Z}})$ is a free $\mathbb{Z}$-module.

Problem 7. Given a Riemann surface X, we let Ω be the presheaf of holomorphic 1-forms on $\mathbb{P}^1$. Its sections on an open chart (U, z) are $\Omega(U) := \{f dz \colon f \in \mathcal{O}(U)\}$. Show that, in fact, Ω is a sheaf on X.

Problem 8 (Forster §13.2)**.** Consider the standard open covering $\mathcal{U} = (U_0 := \mathbb{P}^1 \setminus \{\infty\}, U_\infty := \mathbb{P}^1 \setminus \{0\})$ of $\mathbb{P}^1$.

1. Show that $\mathcal{U}$ is a Leray covering for the sheaf Ω from Problem 7.
2. Prove that $\Omega(U_0 \cap U_\infty) \simeq Z^1(\mathcal{U}, \Omega)$ as vector spaces.
3. Show that $H^1(\mathbb{P}^1, \Omega) = H^1(\mathcal{U}, \Omega)$ is a 1-dimensional vector space, with basis given by the cohomology class of $\frac{dz}{z}$.