MATH 8140 - Topics in Algebraic Geometry (Riemann surfaces)

Homework 6

Divisors, the Riemann-Roch Theorem, linear systems and maps to projective spaces

Please indicate any source in the literature used in finding the solution to a given problem. You are encouraged to work in teams. If you do so, please indicate the name of your collaborator(s) on your submission.

Solutions to each problem can be uploaded (on Carmen) by at most one student. There is no deadline, so work at your own pace.

Please use the following name for the file you upload `HW#_Problem#.pdf`.

Problem 1. Fix a compact Riemann surface X and a divisor D on it. If $\deg(D) < 0$, show that the 0th Čech cohomology group $H^0(X, \mathcal{O}_D)$ vanishes.

Problem 2 (Forster §16.4 (a))**.** Let X be a compact Riemann surface and define the presheaf $\mathcal{D}$ of divisors on X as follows. For each open $U \subset X$ we define $\mathcal{D}(U)$ as the maps $D\colon U \to \mathbb{Z}$ such that for every compact subset $K \subset U$ the Support of D restricted to K is finite. The restriction maps are defined in the natural way.

(i) Show that $\mathcal{D}$ defines a sheaf.

(ii) Show that $H^1(X, \mathcal{D}) = 0$. (*Hint:* Use a discontinuos integer value partition of unity and the techniques used to prove that $H^1(X, \mathcal{E}) = 0$, where $\mathcal{E}$ is the sheaf of differentiable functions on X.)

Problem 3 (Forster §16.4 (b))**.** Fix a compact Riemann surface X and the sheaf of divisors $\mathcal{D}$ on X defined in Problem 2. We define a sequence of maps of sheaves:

$$0 \longrightarrow \mathcal{O}^* \xrightarrow{\alpha} \mathcal{M}^* \xrightarrow{\beta} \mathcal{D} \longrightarrow 0, \tag{1}$$

where $\alpha\colon \mathcal{O}^* \to \mathcal{M}^*$ is the natural inclusion map and $\beta\colon \mathcal{M}^* \to \mathcal{D}$ satisfies $\beta_U(f) = (f) \in \mathcal{D}(U)$ for each $f \in \mathcal{M}^*(U)$ and each open $U \subset X$.

(i) Show that (1) is an exact sequence.

(ii) Compute the connection map $\delta^*\colon Div(X) \to H^1(X, \mathcal{O}^*)$.

Problem 4 (Forster §16.1)**.** Fix a divisor D on $\mathbb{P}^1$. Show:

(i) $\dim H^0(\mathbb{P}^1, \mathcal{O}_D) = \max\{0, 1 + \deg(D)\}$,

(ii) $\dim H^1(\mathbb{P}^1, \mathcal{O}_D) = \max\{0, -1 - \deg(D)\}$.

Problem 5 (Forster §17.4)**.** Fix a compact Riemann surface X of genus g and a divisor D on it. Show:

(i) $0 \leq \dim H^0(X, \mathcal{O}_D) \leq 1 + \deg(D) \quad$ for $-1 \leq \deg(D) \leq g - 1$,

(ii) $1 - g + \deg(D) \leq \dim H^0(X, \mathcal{O}_D) \leq g \quad$ for $g - 1 \leq deg(D) \leq 2g - 1$,

(iii) $\dim H^0(X, \mathcal{O}_D) = 1 - g + \deg(D) \quad$ if $\deg(D) \geq 2g - 1$.

Problem 6 (Forster §17.3). Fix a compact Riemann surface X of genus $g > 0$. Consider the canonical divisor K on X and a divisor $D \geq K$ with $\deg(D) = \deg(K) + 1$. Show that the sheaf $\mathcal{O}_K$ is globally generated by $\mathcal{O}_D$ is not.

Problem 7 (Forster §17.7). Let X be a compact Riemann surface of genus two. Consider a basis $\{\omega_1, \omega_2\}$ of $H^0(X, \Omega)$, and define $f \in \mathcal{M}(X)$ via $\omega_1 = f\,\omega_2$ Show that $f\colon X \to \mathbb{P}^1$ is a degree two holomorphic map.

Problem 8 (Miranda §VII.1.B). Fix a compact Riemann surface of genus $g \geq 2$ an a divisor D on it with $\deg(D) > 0$. Show:

(i) if $\deg(D) \leq 2g - 3$ then $\dim H^0(X, \mathcal{O}_D) \leq g - 1$,

(ii) if $\deg(D) = 2g - 2$ then $\dim H^0(X, \mathcal{O}_D) \leq g$. Deduce from (ii) that among the divisors of degree $2g-2$, the sheaf $\mathcal{O}_D$ with most sections is $\mathcal{O}_K$.

Recall that a divisor D on a compact Riemann surface X is *very ample* if the complete linear system $|D|$ is base point free and $\phi_D\colon X \to \mathbb{P}^N$ is am embedding, where $\dim(H^0(X, \mathcal{O}_D) = N + 1$.

Problem 9 (Miranda §VII.1.C). Fix a compact Riemann surface of genus $g \geq 2$.

(i) Show if $g \geq 3$, then mK is very ample for every $m \geq 2$.

(ii) Show that if $g = 2$, then mK is very ample for every $m \geq 3$.

(iii) Show that if $g = 2$, then $\phi_{2K}\colon X \to \mathbb{P}^2$ is a degree two holomorphic map. (*Hint:* $\dim(H^0(X, \mathcal{O}_{2K}) = 3$ by Problem 5(iii))

(iv) Show that if $g = 2$ the image of ϕ_{2K} is a smooth plane conic curve.