Lecture I: Overview, Basic definitions & examples

§1.1 Overview:

- The construction of Riemann surfaces motivated by issue of $\exists$ holomorphic multivalued functions (eg: $\ln(z)$, $\sqrt{1+z}$, etc.)

  Eg: $\mathbb{C}\setminus\mathbb{R}_{\leq 0} \longrightarrow \mathbb{C}$ can't be extended holomorphically to
  $z \longmapsto \ln z$

  a function $\mathbb{C} \longrightarrow \mathbb{C}$ but $\exists$ X R.S. & $X \xrightarrow{F} \mathbb{C}$ F single valued holo.

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "\pi"'] & \mathbb{C} \\ \mathbb{C}\setminus\mathbb{R}_{\leq 0} \arrow[ur, "\ln z"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

- Locally modelled on open, connected subsets of $\mathbb{C}$. (Assumed connected & Hausdorff)
- Main examples: (1) $\mathbb{C}$, $\mathbb{D} = \{z\in\mathbb{C} : |z|<1\}$, non-compact, genus $=0$

  (2) $\mathbb{P}^1 = \mathbb{C}\mathbb{P}^1$, compact, genus $=0$

  (3) $E_\tau = \mathbb{C}/_{\mathbb{Z}+\tau\mathbb{Z}}$ ($\tau\in\mathbb{C}$ generic), compact ($\underset{\text{homeo}}{\simeq} S^1\times S^1$), genus $=1$

- In this course we'll see 4 ways of building R.S. from old ones

(1) Analytic continuation of local germs of holomorphic functions on X (Weierstrass)

Sheaf $\mathcal{O}$ of holomorphic functions on X $\rightsquigarrow$ $|\mathcal{O}|$ Topological space

$$\begin{array}{cc} X & |\mathcal{O}| \\ \text{R.S.} & \downarrow \pi \\ (\text{eg } \mathbb{P}^1) & X \end{array}$$

φ local germ at x ($\varphi\in\mathcal{O}_x$) $\rightsquigarrow$ $(x,\varphi)\in|\mathcal{O}|$ & Y = connected component of (x,φ)

$\rightsquigarrow$ Y R.S. + a holomorphic map $\begin{array}{c} Y \\ \pi\downarrow \\ X\end{array}$

(2) R.S. associated to holomorphic 1-forms on X. $\rightsquigarrow$ Ω^1: Sheaf on X

(3) [Puiseux approach]: $P(z,w)\in\mathbb{C}(z)[w]$ irreducible polynomial

$\rightsquigarrow$ Algebraic R.S. $\begin{array}{c} Y \text{ "=" } Z(P) \\ \downarrow \\ \mathbb{P}^1 \end{array}$ $\xleftrightarrow{\text{"Galois correspondence"}}$ $\begin{array}{c} \mathcal{M}(Y) = \text{field of meromorph. func on } Y \\ | \quad \text{field extension} \\ \mathcal{M}(\mathbb{P}^1) = \mathbb{C}(z) \end{array}$

(4) X R.S $\rightsquigarrow$ $\tilde{X}$ universal cover is a R.S.

$$\begin{tikzcd} \tilde{X} \arrow[d, "\pi"'] \\ X \end{tikzcd}$$

If $H \subset \mathrm{Aut}_{\mathrm{Conf}}(\tilde{X}) = \{ f : \tilde{X} \to \tilde{X} \ \ f \text{ is biholomorphic} \}$ is a subgroup &

$H \curvearrowright \tilde{X}$ proper discontinuous group action ($\Rightarrow$ with no fixed pts), then

$\tilde{X}/H$ is a R.S.

Theorems:
- $\mathrm{Deck}(\tilde{X}/X) \simeq \pi_1(X)$
- $$\begin{tikzcd} \tilde{X} \arrow[d] \arrow[dr] \arrow[d, phantom, shift left=3ex, "\circlearrowleft"] & \\ Y \arrow[r] & X \end{tikzcd} \quad \longleftrightarrow \quad \begin{array}{l} \pi_1(Y) \lhd \mathrm{Deck}(\tilde{X}/X) \\ \mathrm{Deck}(Y/X) \simeq \mathrm{Deck}(\tilde{X}/X) / \pi_1(Y) \end{array}$$

  RS "topological Galois covering"

  "Galois correspondence"

Main Tools:

- (1) & (4): Topological results (MATH 6801)
  - Covering spaces, path lifting properties,
  - Deck Transformations relation with fundamental groups
  - existence of universal covers.
- (2) & (3): Complex Analytic results (MATH 6221)
  - Properties of holomorphic/meromorphic functions, Cauchy's formula, Residues
  - Identity Theorem, Open Mapping Theorem, Local Behavior of holomorphic functions

(2) Forms & Integration on R.S. Stokes' Theorem. Periods on R.S.

- Holy grail: find non-constant meromorphic functions on R.S.

We'll do this for compact R.S. To do so, we'll develop:
- Sheaves $\mathcal{F}$ on RS & Čech cohomology:

- covering dependent def $H^p(\underline{\mathcal{U}}, \mathcal{F}) \rightsquigarrow$ take a direct limit $H^p(X, \mathcal{F}) = \varinjlim_{\underline{\mathcal{U}}} H^p(\underline{\mathcal{U}}, \mathcal{F})$

- Leray's Theorem (X compact R.S.): $H^1(X,\mathcal{F}) = H(\underline{\mathcal{U}},\mathcal{F})$ if $\underline{\mathcal{U}}$ is nice enough $(H^1(\mathcal{U}_i,\mathcal{F})=0 \ \forall i)$
- Finiteness result: If X is compact, $H^1(X,\mathcal{O})$ is finite dim'l $\mathbb{C}$-v.sp.

Theorems: - $\exists$ non-constant meromorphic functions on compact R.S.
- compact R.S. are algebraic

- Divisors on X, Canonical divisors, sheaves $\mathcal{O}_D$, Ω_D (X: compact R.S.)
- algebraic characterization of genus of X (compact) $\dim H^1(X,\mathcal{O})$
- 2·Top genus of X $= \mathrm{rk}\, H_1(X,\mathbb{Z}) = \dim_{\mathbb{R}} H^1_{dR}(X) = \dim_{\mathbb{C}} H^1_{dR}(X,\mathbb{C}) = \dim_{\mathbb{C}} H^1(X,\underline{\mathbb{C}}) = 2g$
- Riemann-Roch Theorem: $\dim H^0(X,\mathcal{O}_D) - \dim H^1(X,\Omega_D) - \deg D$ is indep. of the divisor D $(= 1-g)$
- Serre duality: $\Omega_{-D}(X) \simeq H^1(X,\mathcal{O}_D)^\vee$ (dual vector space)
- Riemann-Hurwitz formula: relating genera of n-sheeted cover $Y \longrightarrow X$ between compact R.S.

- Abel-Jacobi Theory on compact R.S.
  - Q: which divisors come from meromorphic functions?
  - X compact R.S., genus $g \geq 1 \rightsquigarrow \mathrm{Jac}(X) = \mathbb{C}^g/\Lambda_{2g}$ ab var + $X \xrightarrow[\text{Abel-Jacobi}]{} \mathrm{Jac}(X)$

    Λ_{2g} ↳ lattice of rk $2g$ (period lattice)

  - $X^g \twoheadrightarrow \mathrm{Jac}(X)$ via Periods $\longleftrightarrow$ $\mathrm{Pic}(X) = \dfrac{\mathrm{Div}_0(X)}{\mathrm{Ppal\,Div}(X)} \simeq \mathrm{Jac}(X)$

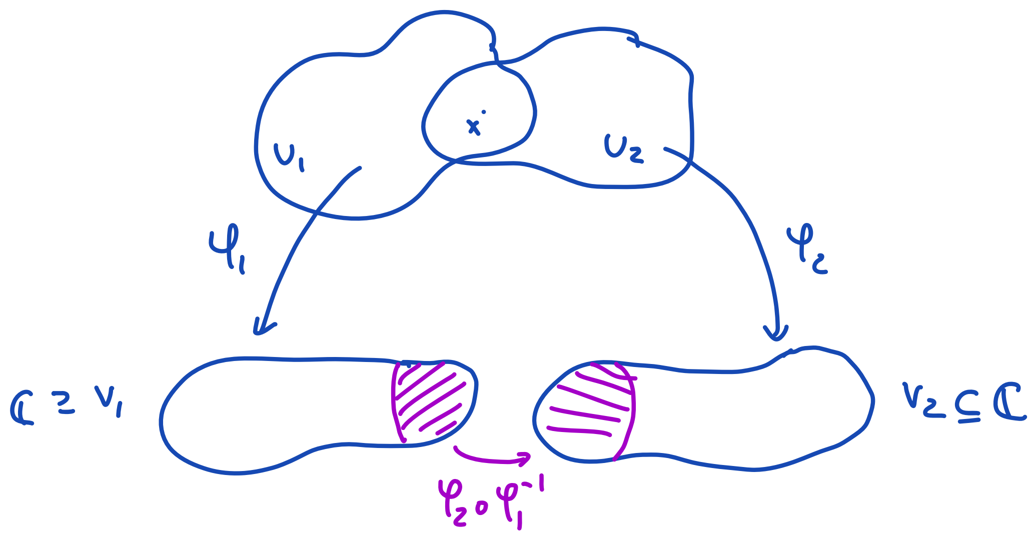
- Extra Topics?
  - Algebraic proofs of some results
  - Applications of Riemann Roch (canonical embeddings, inflection & Weierstrass pts)
  - Hurwitz Theory (count $Y \longrightarrow X$ maps between compact R.S. with prescribed ramification profile.)
  - Uniformization Thm (only simply connected R.S. are $\mathbb{C}$, $\mathbb{D}$ or $\mathbb{P}^1$)

§1.2 The definition of a R.S:

INPUT: A 2-dim'l manifold $\boxed{X}$, i.e. a connected Hausdorff topological space X s.t. $\forall x \in X$ $\exists\, \mathcal{U} \underset{\text{open}}{\subset} X$ with $x \in \mathcal{U}$ st $\mathcal{U}$ is homeomorphic to an open set in $\mathbb{R}^2$

Definition: A complex chart on X is a homeomorphism $\varphi: U \longrightarrow V$ of an open subset $U \subset X$ onto an open subset $V \subseteq \mathbb{C}$

- Two charts $\varphi_i : U_i \longrightarrow V_i$ for $i = 1, 2$ are holomorphically compatible if the map $\quad \varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \longrightarrow \varphi_2(U_1 \cap U_2)$ is biholomorphic



- A complex atlas on X is a system $\mathcal{U} = \{\varphi_i : U_i \rightarrow V_i\}_{i \in I}$ of holomorphically compatible charts with $X = \bigcup_{i \in I} U_i$. It defines a complex structure on X

Def: A Riemann Surface (R.S.) is a pair (X, Σ) where X is a (connected) 2D manifold and Σ is a complex structure on X.

Remarks: • Two complex atlases $\mathcal{U}, \mathcal{U}'$ are analytically equivalent if $\mathcal{U} \cup \mathcal{U}'$ is a complex atlas. They define equivalent complex structures on X, ie the same R.S.

- The last notion is an equivalence relation on complex atlases on X
- We can define a poset structure on complex atlases. Every atlas $\mathcal{U}$ has a unique max'l atlas $\mathcal{U}^*$ refining it. This is often used when defining Σ

Remark: Can always pick charts with $\varphi: \mathcal{U} \xrightarrow{\sim} \mathbb{D} = \{z : |z|<1\}$ (coord. nbhd)

$x \in \mathcal{U}$, $x \xmapsto{z} 0$

⚠ Some X can be endowed with different complex structures.

Eg: X compact $\rightsquigarrow$ X is determined by its topological genus g
($\simeq S^2$ with g handles glued on it)

But if $g=1$ $X \simeq S^1 \times S^1$ (complex torus). (1 topology)

Genus one sm. algebraic curves $/\mathbb{C}$ are completely determined by their j-invariant $\rightsquigarrow$ moduli space $\mathcal{M}_1$ of complex structures on X (1-dimensional)

§1.3 Examples:

(1) $X = \mathbb{C}$, $\Sigma: \mathbb{C} \xrightarrow{id} \mathbb{C}$

(2) $\mathcal{U} \subseteq \mathbb{C}$ open connected set in $\mathbb{C}$ (eg $\mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$, $\mathbb{C} \smallsetminus \mathbb{R}_{\leq 0}$)

(3) $\mathbb{CP}^1 = \mathbb{P}^1 = \hat{\mathbb{C}}$ Riemann sphere / projective line.

$\mathcal{U}_0 = \{(x,y) : x \neq 0\}/\sim \longleftrightarrow \frac{y}{x} \in \mathbb{C} \simeq \mathbb{R}^2$ $\quad$ $\mathcal{U}_\infty = \{(x,y) : y \neq 0\}/\sim \longleftrightarrow \frac{x}{y} \in \mathbb{C} \simeq \mathbb{R}^2$

$\rightsquigarrow$ • $\mathbb{P}^1$ is a 2D manifold
• $\mathcal{U}_0 \cap \mathcal{U}_\infty$ is connected $\Rightarrow$ $\mathbb{P}^1$ is connected

• Complex structure:

$$\mathcal{U}_0 = \mathbb{P}^1 \smallsetminus \{\infty\} \xrightarrow{\varphi_0} \mathbb{C}$$
$$\{(1,z) : z \in \mathbb{C}\} \longrightarrow z$$

$$\mathcal{U}_\infty = \mathbb{P}^1 \smallsetminus \{0\} \xrightarrow{\varphi_\infty} \mathbb{C}$$
$$\{(\omega,1) : \omega \in \mathbb{C}\} \longrightarrow \omega$$

$\mathbb{C} \longrightarrow \mathbb{C}$
$z \longmapsto z$

(complex charts)

$\mathbb{C}^\times \cup \{\infty\} \longrightarrow \mathbb{C}$
$z \longmapsto \begin{cases} \frac{1}{z} & z \in \mathbb{C}^\times \\ 0 & z = \infty \end{cases}$

$$\varphi_\infty \circ \varphi_0^{-1} : \varphi_0(\mathcal{U}_0 \cap \mathcal{U}_\infty) = \mathbb{C}^\times \longrightarrow \mathbb{C}^\times = \varphi_\infty(\mathcal{U}_0 \cap \mathcal{U}_\infty) \quad \text{is biholomorphic}$$
$$z \longrightarrow \frac{1}{z}$$

Note: $\mathbb{P}^1$ is compact ($\mathbb{C} \cup \{\infty\}$ with 1-pt compactification $\simeq S^2 \subseteq \mathbb{R}^3$ via stereographic projection)

(4) Pick $\omega_1, \omega_2 \in \mathbb{C}^\times$ st $\frac{\omega_2}{\omega_1} \notin \mathbb{R}$ $(\{\omega_1, \omega_2\}$ l.i over $\mathbb{R})$

- $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subseteq \mathbb{C}$ is a discrete lattice

$(\exists\, \varepsilon > 0$ st $|z| > \varepsilon$ if $z \in \Gamma \setminus \{0\})$

- z, z' in $\mathbb{C}$ are Λ-equivalent if $z - z' \in \Gamma$

Consider $E = \mathbb{C}/\Gamma$ with the quotient Topology induced by π $\begin{array}{c}\mathbb{C}\\ \downarrow\\ E=\mathbb{C}/\Gamma\end{array}$

$(U \subset E$ is open $\iff \pi^{-1}(U) \subseteq \mathbb{C}$ is open$)$ So π is continuous & thus, E is connected.

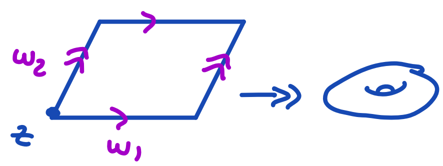
- π is an open map: Fix $V \subseteq \mathbb{C}$ open

$\pi(V) \subseteq E$ is open $\iff \pi^{-1}(\pi(V)) \subseteq \mathbb{C}$ is open

But $\pi^{-1}(\pi(V)) = \bigcup_{p \in \Lambda} \underbrace{(p + V)}_{\text{open}} \subseteq \mathbb{C}$ is open.

- E is a Hausdorff Top space because Λ is discrete. $(\pi(D(z_0, \varepsilon))$ separate pts in $E)$
- E is compact:

$z \in \mathbb{C} \rightsquigarrow \mathcal{B}_z = \{ z + \lambda_1 \omega_1 + \lambda_2 \omega_2 : \lambda_i \in [0,1] \}$



By construction $\mathcal{B}_z$ is compact in $\mathbb{C}$ & $\pi(\mathcal{B}_z) = E$, so E is compact

- Complex structure:

For each $x \in E$, Pick $v \in \pi^{-1}(x)$ & $v \in V \subseteq \mathbb{C}$ open st $\pi|_V$ is injective.

Then $U = \pi(V)$ is open in E, $x \in E$ & $\pi|_V : V \to U$ is homeo.

$\Rightarrow$ Define $\varphi = (\pi|_V)^{-1} : U \to V$ as our complex chart.

- These charts are holomorphically compatible. (Exercise. Use Λ is discrete)

Exercise: $\mathbb{C} = \mathbb{R}\omega_1 \oplus \mathbb{R}\omega_2 \longrightarrow S^1 \times S^1$ factors through $\mathbb{C}/\Lambda$

$\lambda_1\omega_1 + \lambda_2\omega_2 \longmapsto (e^{2\pi i \lambda_1}, e^{2\pi i \lambda_2})$

& gives a homeomorphism $\mathbb{C}/\Lambda \simeq S^1 \times S^1$ (complex Torus)