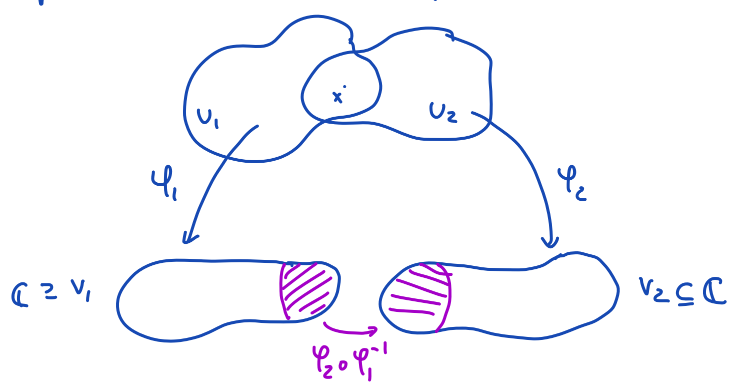
Lecture II : Holomorphic / Meromorphic maps on R.S.

Recall A Riemann surface is a pair (X, Σ) where :

- X is a 2D real manifold that is Hausdorff & connected
- Σ is a complex structure on X, ie a (max'l) collection of holomorphically compatible complex charts $\{(\underset{\text{open}}{U_i}, \varphi_i : U_i \xrightarrow[\text{homeo}]{} \underset{\text{open}}{V_i} \subseteq \mathbb{C})\}_{i\in I}$ with $X = \bigcup_{i\in I} U_i$.

φ_1 & φ_2 are holo compatible if $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \longrightarrow \varphi_2(U_1\cap U_2)$ is biholomorphic



Note: Can take $V_i = \mathbb{D}$ a disc centered at 0. (chart is a "coordinate nbhd")

Examples: $\mathbb{C}$, $\mathbb{C}^\times$, $\mathbb{P}^1$, $E = \mathbb{C}/\Lambda$ $\quad \Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ with $\{\omega_1, \omega_2\}$ $\mathbb{R}$-basis for $\mathbb{C}$.

Q: Which lattices give the same R.S.?

Answer involves conformally equivalent lattices. (later in the course)

TODAY: Holomorphic/meromorphic on R.S & holomorphic functions between R.S.

§ 2.1 Holomorphic/ Meromorphic functions on $\mathbb{C}$:

Model: $f : \mathcal{U} \longrightarrow \mathbb{C}$ holo / meromorphic function on $\mathcal{U} \subseteq \mathbb{C}$ open & connected

- f is $\mathbb{C}$-differentiable $\Longleftrightarrow$ $f_{(x+iy)} = u_{(x,y)} + i\, v_{(x,y)}$ with u, v differentiable & satisfying the Cauchy-Riemann Equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

($\Rightarrow$ u, v are harmonic, ie $u_{xx} + u_{yy} = 0$)
if u, v are C^2-functions

Key Property: Holomorphic functions are analytic (near each $z_0 \in \mathcal{U}$ $f_{(z)}$ can be expressed as a power series in $(z-z_0)$ with a positive radius of convergence $\geq d(z_0, \partial \mathcal{U})$)

Identity Theorem: Fix $\mathcal{U} \subseteq \mathbb{C}$ open & connected & let $f, g : \mathcal{U} \to \mathbb{C}$ be two holomorphic functions. Suppose $\exists A \subseteq \mathcal{U}$ infinite set with a limit pt in $\mathcal{U}$ with $f_{(a)} = g_{(a)}$ $\forall a \in A$. Then, $f = g$.

Open Mapping Theorem: If $f : \mathcal{U} \to \mathbb{C}$ is holomorphic, then $f(\mathcal{U}) \subseteq \mathbb{C}$ is open. In particular, f is an open map.

- Say $\mathcal{U} = D(z_0, R)^* = D(z_0, R) \setminus \{z_0\}$ & $f : \mathcal{U} \to \mathbb{C}$ holomorphic

Def: z_0 is an isolated singularity of f

Q: What is the behavior of f near z_0?

(1) Removable Singularity at z_0: $\lim_{z \to z_0} (z-z_0) f_{(z)} = 0$.

Equivalently: if f is bounded on $\mathcal{U}$. ($\exists M > 0$ st $|f_{(z)}| \leq M$ $\forall z \in \mathcal{U}$)

In particular, f can be extended holomorphically to z_0 via $\lim_{z \to z_0} f_{(z)}$.

(2) Pole at z_0:

$\lim_{z \to z_0} |f_{(z)}| = \infty \implies f$ has a pole at z_0, ie $\exists k > 0$ with $g_{(z)} = (z-z_0)^k f_{(z)}$ holomorphic at z_0.

So $f_{(z)}$ can be expressed as a Laurent power series in $(z-z_0)$, ie

$$f_{(z)} = \sum_{\ell \geq -k} a_k (z-z_0)^\ell \quad \text{with } a_k \neq 0$$

$\boxed{k = \operatorname{ord}_{z_0}(f)}$ = order of z_0 as a pole of f.

Equivalently: $f(z) = \frac{A_k}{(z-z_0)^k} + \cdots + \frac{A_1}{(z-z_0)} + f_1(z)$ &

$f_1(z)$ is holomorphic on $D(z_0,R)$ & $A_k \neq 0$.

(3) Essential singularity at z_0:

$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ & $a_n \neq 0$ for infinitely many $n<0$.

Casorati-Weierstrass Thm: If f has an isolated essential singularity at z_0

then $\forall \delta > 0$: $\overline{f(D^*(a,\delta))} = \mathbb{C}$

Note: Non-isolated singularities satisfy $\nexists \lim_{z\to z_0} |f(z)|$ (we can think of non-isolated as also essential singularities).

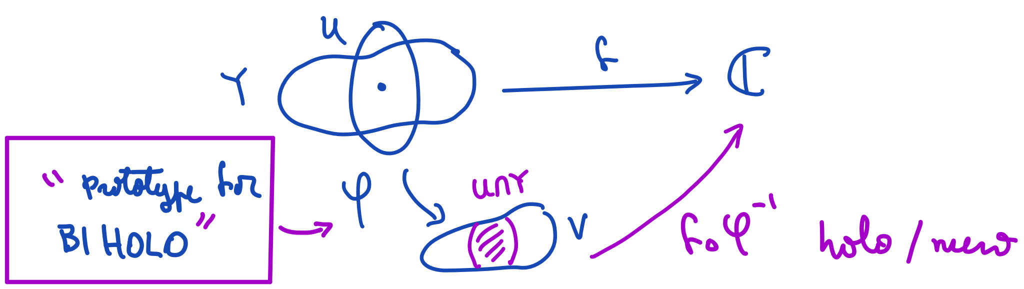
Def: f is meromorphic if all its singularities are poles (by def, isolated)

Note: Poles of a meromorphic function can accumulate (at a (non-isolated) essential sing)

Def: $f: U \longrightarrow V$, $U,V \subseteq \mathbb{C}$ opens is biholomorphic if f bijective with f & f^{-1} both holomorphic

§ 2.2 Holomorphic/ Meromorphic functions on R.S.: Fix X: R.S.

Def Fix $Y \subset X$ open. A function $f: Y \longrightarrow \mathbb{C}$ is holomorphic / meromorphic if for every chart $\varphi: U \longrightarrow V \subseteq \mathbb{C}$ on X, the map $f\circ\varphi^{-1}: \varphi(U\cap Y) \longrightarrow \mathbb{C}$ is holomorphic / meromorphic



Exercise: Show this defn is atlas independent, e.g. by working with mxl charts.

Alternative definition $f : Y \dashrightarrow \mathbb{C}$ is **meromorphic** if $\exists\, P \subset Y$ discrete st. $f : Y \setminus P \longrightarrow \mathbb{C}$ is holomorphic, &

- $\forall p \in P \ \lim\limits_{\substack{y \to p \\ y \in Y \setminus P}} |f(y)| = \infty$ (equiv. $\varphi(U \cap P)$ has a nbhd along which $f \circ \varphi^{-1}$ admits a Laurent series expansion at p)

Notation: $\mathcal{O}(Y)$ = ring of holomorphic functions on Y. $\rightsquigarrow$ sheaf $\mathcal{O}$ on X

$\mathcal{M}(Y)$ = ______ meromorphic __________ $\rightsquigarrow$ sheaf $\mathcal{M}$ on X

Remark:
- $\mathcal{O}(Y) \subseteq \mathcal{M}(Y)$ is a subring. Both are $\mathbb{C}$-algebras
- $\mathcal{M}(Y)$ is a field $\iff$ Y is connected.

(**Reason:** Zeros & Poles of non-constant meromorphic functions are isolated)

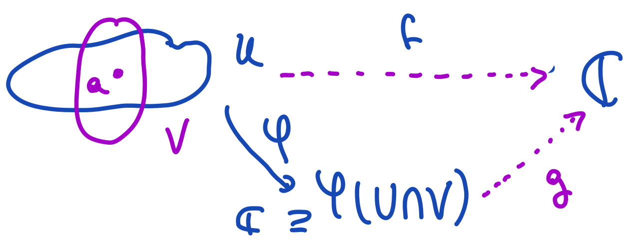
Removable Singularity Theorem: Fix X a R.S, $Y \subseteq X$ open & $a \in Y$

Pick $f \in \mathcal{O}(Y \setminus \{a\})$ & assume f is **bounded near a** (ie $\exists\, V \subseteq Y$ open with $a \in V$ & $M \in \mathbb{R}_{>0}$ st $|f(x)| < M \ \forall x \in V \setminus \{a\}$)

Then, f extends to a holomorphic function on Y.

Pf/ Let (U, φ) be a chart with $a \in U \subseteq Y$ & write $D = \varphi(U \cap V)$ with $z_0 = \varphi(a)$. Then $g = f \circ \varphi^{-1} : D \setminus \{z_0\} \longrightarrow \mathbb{C}$ is a bounded holomorphic function.

By the Removable Sing Thm for $\mathbb{C}$, $w = \lim\limits_{z \to z_0} g(z)$ exists & this is the value assigned to $f(a)$. Moreover, g is holomorphic on D

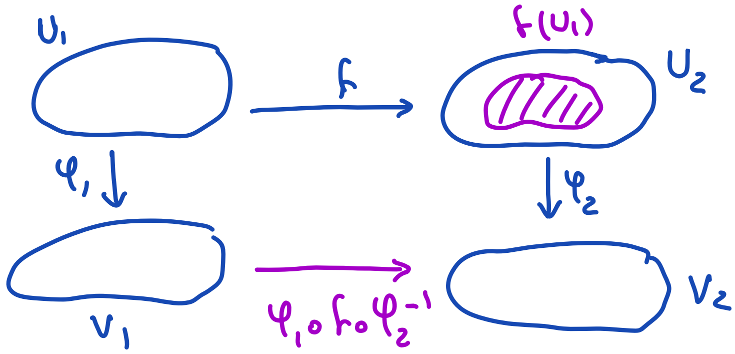


g has a removable sing at a. □

§2.3 Holomorphic functions between R.S.:

Fix X_1, X_2 R.S.

Definition: A continuous function $f: X_1 \longrightarrow X_2$ is holomorphic if for every pair of open charts $(U_1, \varphi_1 : U_1 \longrightarrow V_1)$ & $(U_2, \varphi_2 : U_2 \longrightarrow V_2)$ in maxl atlases for X_1 & X_2 with $f(U_1) \subseteq U_2$ we have $\varphi_2 \circ f \circ \varphi_1^{-1} : V_1 \longrightarrow V_2$ is holo



Observations: (1) This definition extends $f: X \longrightarrow \mathbb{C}$ holomorphic.

(2) Compositions of holomorphic functions between R.S are holomorphic.

Proposition: $f: X_1 \longrightarrow X_2$ continuous is holomorphic iff for every $Y_2 \subseteq X_2$ open & $g \in \mathcal{O}(Y_2)$: $g \circ f : f^{-1}(Y_2) \longrightarrow \mathbb{C}$ is holomorphic.

Pf: Exercise

- This result allows us to define the notion of pullbacks of holomorphic functions between R.S.

$f: X_1 \longrightarrow X_2$ holo $\rightsquigarrow$ $f^* : \mathcal{O}(Y_2) \longrightarrow \mathcal{O}(f^{-1}(Y_2))$

$Y_2 \subseteq X_2$ open $\qquad g \longmapsto g \circ f$

Note: • f^* is a ring homomorphism

• $(h \circ f)^* = f^* \circ h^*$

Definition: • A map $f: X_1 \longrightarrow X_2$ is biholomorphic if it is bijective and both f & f^{-1} are holomorphic.

• Two R.S. X_1, X_2 are isomorphic if $\exists\, f: X_1 \longrightarrow X_2$ biholomorphism.

Identity Theorem: Fix X_1 & X_2 R.S & $f, g: X_1 \longrightarrow X_2$ holomorphic. Assume $\exists\, A \subset X_1$ infinite with an accumulate pt & $f(a) = g(a)$ $\forall a \in A$. Then, $f = g$.

Pf/ Fix a_0 = a limit pt of A, take charts around a_0 & $f(a_0)$ & use the $\mathbb{C}$-analytic result.

Alternatively, define $G = \{x \in X_1 : \exists$ open set $U \subseteq X$ with $x \in U$ & $f|_U = g|_U\}$

• G is open: by construction.

• G is closed: Pick $x \in \partial G$ & a sequence $(x_n) \subseteq G$ with $x_n \to x$.

Since $f(x_n) = g(x_n)$ $\forall n$ & f, g are cont then $f(x) = g(x)$.

Let (U_1, φ_1) be a coordinate nbhd in X_1 with $x \in U_1$ &

(U_2, φ_2) ——— X_2 with $f(U_1) \subseteq U_2$, $g(U_1) \subseteq U_2$

$$\begin{tikzcd} U_1 \arrow[r, "\varphi_1", "\sim"'] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & \mathbb{D} \subseteq \mathbb{C} \arrow[d, shift right] \arrow[d, shift left, "\varphi_2\circ f\circ\varphi_1 = \varphi_2\circ g\circ\varphi_2"] \\ U_2 \arrow[r, "\varphi_2", "\sim"'] & \mathbb{D} \subseteq \mathbb{C} \end{tikzcd}$$

(by Id Thm on $\mathbb{C}$ applied to $A = \{x_n\}_{n\in\mathbb{N}}$)

In particular, $f|_{U_1} = g|_{U_1}$ & $x \in U_1$ open, so $x \in G$.

• Same logic says all accumulation pts of A are in G., so $G \neq \emptyset$

Since X_1 is connected & $G \neq \emptyset$ is open & closed, we get $G = X_1$ □

Example. Fix $n \geq 1$ & $F(z) = z^n + c_1 z^{n-1} + \dots + c_0 \in \mathbb{C}[z]$

Then, F defines a holomorphic function $f : \mathbb{C} \to \mathbb{C}$

Since $\lim_{z \to \infty} |f(z)| = \infty$, then we have $f \in \mathcal{M}(\mathbb{P}^1)$

- Next, we reinterpret $\mathcal{M}(X)$ as $\{h : X \to \mathbb{P}^1 \text{ holomorphic}\}$

How? Fix X & $f \in \mathcal{M}(X)$. Let P be the set of poles of f

Define $\tilde{f} : X \to \mathbb{P}^1$ via
$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \notin P \\ \infty & \text{if } x \in P \end{cases}$$

Theorem: If $f \in \mathcal{M}(X)$, then $\tilde{f} : X \to \mathbb{P}^1$ is holomorphic.

Conversely, if $h : X \to \mathbb{P}^1$ is holomorphic, then $h(x) = \infty$ $\forall x \in X$ or $h^{-1}(\infty)$ is discrete, & $\exists f \in \mathcal{M}(X)$ with $\tilde{f} = h$.

Pf/($\Rightarrow$) $\tilde{f}$ is continuous by construction:

- $\tilde{f}$ is holomorphic on $X \setminus P$
- Pick $p \in P$ & a coordinate chart $(U, \varphi : U \xrightarrow{\sim} \mathbb{D})$ with $p \in U$ & $\varphi(p) = 0$

p is a pole of f, so we can assume $\tilde{f}(U) \subseteq \mathcal{U}_\infty = \mathbb{P}^1 \setminus \{0\}$

Then
$$\begin{tikzcd} U \arrow[r, "\sim", "\varphi"'] \arrow[d, "\tilde{f}"] & \mathbb{D} \\ \mathcal{U}_\infty \arrow[r, "\sim", "\varphi_\infty"'] & \mathbb{C} \\ z \arrow[r, mapsto] & \frac{1}{z} \end{tikzcd}$$
$\Rightarrow$ $\mathbb{D}^* \xrightarrow{\varphi_\infty \circ \tilde{f} \circ \varphi^{-1}} \mathbb{C}$ has a removable singularity at $z_0 = 0 = \varphi(p)$ because it is bounded near 0.

$\Rightarrow$ $\varphi_\infty \circ \tilde{f} \circ \varphi^{-1} \in \mathcal{O}(\mathbb{D})$, ie $\tilde{f}|_{\mathcal{U}}$ is holomorphic

($\Leftarrow$) Assume h is not constantly ∞.

- $h^{-1}(\infty)$ is closed so $X' = X \setminus h^{-1}(\infty)$ is open. & $h|_{X'}$ is holo. (holomorphic map on each connected component of X').
- Pts in $h^{-1}(\infty)$ are isolated by the Identity Theorem on holomorphic maps between R.S.
- Set $f = h|_{X \setminus h^{-1}(\infty)}: X \setminus h^{-1}(\infty) \longrightarrow \mathbb{C}$. It's holomorphic by construction.

Since h is continuous, we get $\displaystyle\lim_{\substack{x \in X' \\ x \to p}} |f(x)| = \infty$ for each $p \in h^{-1}(\infty)$,

so f is meromorphic & $h^{-1}(\infty) =$ poles of f. $\square$

<u>Exercise</u>: Show $\mathcal{M}(\mathbb{P}^1) = \mathbb{C}(z)$.

The following results follow from the definition of holomorphic functions between R.S. + results from $\mathbb{C}$-analysis from §2.1.

<u>Corollary</u> 1 (Open Mapping Thm) $f: X_1 \longrightarrow X_2$ holomorphic is open

<u>Corollary</u> 2 (Maximum Principle)

If $f: X \to \mathbb{C}$ is non-constant & holomorphic, then $|f(x)|$ cannot attain its maximum value on X. f is

Pf/ $f(X)$ is open so if $|f(a)| = \sup_{x \in X} |f(x)|$ $\exists\, \varepsilon > 0$ s.t $D(f(a), \varepsilon) \subseteq f(X)$ so $|f(a)|$ was not max. Contr!

<u>Corollary 3</u>: Suppose X, Y are R.S & X is compact & $f: X \to Y$ is holomorphic. Then, either (1) f is constant or, (2) f is surjective & Y is compact

Pf/ If f is non-constant, then f is open So $f(X)$ is open

But $f(X)$ is compact in Y so its closed (Y is Hausdorff).

Since Y is connected & $f(X) \neq \emptyset$ is open & closed in Y, we get

$f(X) = Y$ & Y is compact. $\square$

Corollary 4: If X is R.S & $f: X \to \mathbb{C}$ is holomorphic, then f is constant.

§2.4 Local behavior of holomorphic maps:

We'll need the following two results from $\mathbb{C}$-analysis:

Inverse Function Theorem: Assume $U \subseteq \mathbb{C}$ open & connected & fix $a \in U$. Let $f: U \to \mathbb{C}$ be holomorphic with $f'(a) \neq 0$. Then, there exists $r_1, r_2 > 0$ with $D(a, r_1) \subseteq U$ s.t. $f|_{D(a,r_1)}: D(a, r_1) \longrightarrow D(f(a), r_2)$ is biholomorphic.

PF idea: After translation & dilation by $f'(a)$, we assume $a = 0$, $f(a) = 0$ & $f'(a) = 1$

Pick a power series expansion of f around 0 so $f(z) = z + c_2 z^2 + \cdots$

We propose a formal power series $g = z + b_2 z^2 + \cdots$ with $f \circ g = \mathrm{id}$.

Hard part: Show $\mathrm{roc}(g) > 0$ $\square$

More generally:

Local behavior If $U \subseteq \mathbb{C}$ open, connected with $a \in U$, $f: U \to \mathbb{C}$ holomorphic and $f'(a) = \cdots = f^{(N-1)}(a) = 0$ but $f^{(N)}(a) \neq 0$. Then $\exists\, r_1, r_2 > 0$ with $D(a, r_1) \subseteq U$ s.t. $f|_{D(a,r_1)}: D(a, r_1) \longrightarrow D(f(a), r_2)$ is N-to-1 on the punctured discs.

Moreover, we can reparameterize things so that f locally looks like $z \longmapsto z^N$.

Proof: Assume $a = f(a) = 0$ & $f'(a) = 1$ & Take Taylor series expansion of f around 0.

$$f(z) = z^N + c_{N+1} z^{N+1} + \cdots$$

GOAL: Write f as g^N for some power series g with $g(0) = 0$ & $g'(0) \neq 0$.

$g(z) = b_1 z + b_2 z^2 + \cdots$ $\qquad b_1 = g'(0)$ & $b_1^N = 1$.

We have N solutions for $b_1 = N^{th}$ root of 1. Once b_1 is fixed, the rest of the b_i's are uniquely determined.

Why? $f = z^N(1 + c_{N+1} z + \cdots\cdots) = z^N\left(b_1\left(1 + \frac{b_2}{b_1} z + \cdots\cdots\right)\right)^N$

Cancel z^N on both sides to get

$$1 + c_{N+1} z + \cdots\cdots = \left(1 + \frac{b_2}{b_1} z + \cdots\cdots\right)^N = (1 + h(z))^N$$

Take log: $\ln(1 + c_{N+1} z + \cdots\cdots) = N \log(1 + h(z))$

holomorphic around 0

$\Rightarrow$ the series for $\log(1+h(z))$ is uniquely determine by the series on the (LHS) which has $roc = r_1 > 0$.

Take exponential to get the series for $1 + h(z)$ & hence for $h(z)$.

$\Rightarrow f(z) = (z(1+h(z)))^N$ & $S(z) = z(1+h(z))$ restricts to a biholomorphism $S|_{D(0,r_1)} : D(0,r_1) \longrightarrow D(0,r_2)$ by the inverse function Thm.

$\Rightarrow$ Locally, f looks like $z \longmapsto z^N$ which satisfies the N-to-1 condition. $\square$

This local behavior extends to non-constant holomorphic maps between R.S!

<u>Theorem</u>: Fix X_1, X_2 R.S. & $f: X_1 \longrightarrow X_2$ a non-constant holomorphic map. Fix $a_1 \in X_1$ & $a_2 = f(a_1)$. Then $\exists\ N \geq 1$ integer & coordinate nbhds $(U_1, \varphi_1 : U_1 \xrightarrow{\sim} \mathbb{D})$ & $(U_2 : \varphi_2 : U_2 \xrightarrow{\sim} \mathbb{D})$ of a_1 & a_2 respectively

st: (1) $f(U_1) = U_2$

(2) $\varphi_1(a_1) = \varphi_2(a_2) = 0$

(3)
$$\begin{tikzcd} U_1 \arrow[r, "f"] \arrow[d, "\varphi_1"'] \arrow[dr, phantom, "\circlearrowleft"] & U_2 \arrow[d, "\varphi_2"] \\ \mathbb{D} \arrow[r, "g"'] & \mathbb{D} \end{tikzcd}$$

with $g = \varphi_2 \circ f \circ \varphi_1^{-1} : z \longmapsto z^k$.

So $f : U_1 \setminus \{a_1\} \longrightarrow U_2 \setminus \{a_2\}$ is N-to-1.

Furthermore, N is independent of the choice of charts.

Pf/. Conditions (1) & (2) are easy to achieve since f is open.

[Pick $a_2 \in U_2' \xrightarrow{\varphi_2} V_2$ with $U_2' \subseteq f(X)$. Take $D_2 = D(\varphi(a_2), r) \subseteq V_2$ & $U_2 = \varphi_2^{-1}(D_2)$

Then $\psi_2 : U_2 \underset{\varphi_2|_{U_2'}}{\xrightarrow{\sim}} D_2 \underset{\text{biholo}}{\xrightarrow{\cdot 1/r}} D(\varphi(a_2), 1) \underset{\text{biholo}}{\xrightarrow{\sim}} D(0,1)$ gives the chart.

$z \longmapsto z - \varphi_2(a_2)$

Do the same with a chart (U_1', φ_1) around a_1 with $U_1' \subseteq f^{-1}(U_2')$.]

• For (3) consider $g : \mathbb{D} \longrightarrow \mathbb{D}$ <u>Note:</u> $g(0) = 0$ and g is not constant

Pick $N \geq 0$ with $g(0) = g^{(1)}(0) = \cdots = g^{(N-1)}(0) = 0$ & $g^{(N)}(0) \neq 0$.

By the classical result: $\exists$ discs $D(0,r_1)$ & $D(0,r_2)$

$$g : D^*(0,r_1) \longrightarrow D^*(0,r_2) \quad \text{is } N\text{-to-}1.$$

Moreover,

$g = (z\,h(z))^N$ & $s(z) = z\,h(z) : D(0,r_1) \longrightarrow D(0, r_2^{1/N})$ is biholomorphic

We shrink U_1, U_2 so that $\varphi_1(U_1) \subseteq D(0,r_1)$, $\varphi_2(U_2) \subseteq D(0,r_2)$ & $f(U_1) \subseteq U_2$

$$\begin{tikzcd}
U_1 \arrow[r, "\varphi_1"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & D(0,r_1) \arrow[r, "s(z)"] \arrow[d, "g"', "s(z)^N"] \arrow[dr, phantom, "\circlearrowright"] & D(0, r_2^{1/N}) \simeq \mathbb{D} \arrow[dl, "z \mapsto z^N"] \\
U_2 \arrow[r, "\varphi_2"'] & D(0,r_2) \simeq \mathbb{D} &
\end{tikzcd}$$

$\square$

<u>Corollary 1.</u> $\forall\, a_1 \in X_1$ & $a_1 \in U_0 \subseteq X_1$ (open), $\exists$ $U \subseteq U_0$ nbhd of a_1 & W of $a_2 = f(a_1)$ s.t. $f^{-1}(y) \cap U$ has precisely N elements for all $y \in W \setminus \{a_2\}$

N = branching number of f near a_1 = multiplicity of f at a_1.

<u>Exercise:</u> If $f(z) = z^N + c_1 z^{N-1} + \cdots + c_0$ then $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is holo & $f(\infty) = \infty$. Show $\operatorname{mult}(f; \infty) = N$

Corollary 2: If $f: X_1 \longrightarrow X_2$ is holomorphic and injective, then
$f: X_1 \longrightarrow f(X_1)$ is biholomorphic.

Pf/ Branching number is 1 by injectivity, $f(X_1)$ is open & f is locally of the form $z \longrightarrow z$, so its inverse is also holomorphic.