Lecture III: Homotopy of curves, Fundamental group

In these notes we review basics point set topology that we'll need to build new Riemann surfaces from old ones.

§3.1 Curves & homotopies of curves:

Throughout we fix a topological space X and denote the interval $[0,1]$ by I.

Def: A curve on X is a continuous map $u:[0,1] \longrightarrow X$

Names: $u(0)$ = starting pt of the curve ; $u(1)$ = ending pt of the curve

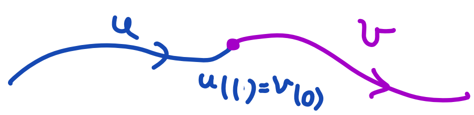
Curves can be "multiplied" and reversed :

Def: Given $u:[0,1] \longrightarrow X$, $v:[0,1] \longrightarrow X$ with $u(1)=v(0)$, then

(1) The product curve $u \cdot v : [0,1] \longrightarrow X$ from $u(0)$ to $v(1)$ is defined by

$$(u \cdot v)(t) = \begin{cases} u(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ v(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

$u \cdot v$ =



We call this operation concatenation.

(2) The inverse curve $u^- = u^{op} : [0,1] \longrightarrow X$ from $u(1)$ to $u(0)$ is defined by

$$u^-(t) = u(1-t) \quad \forall t \in I.$$

$a = u(0)$
$b = u(1)$

(3) The constant curve associated to $a \in X$ is defined as

$$1_a(t) = a \quad \forall t \in [0,1].$$

Note: $(u^-)^- = u$ & $(u \cdot v)^- = v^- \cdot u^-$

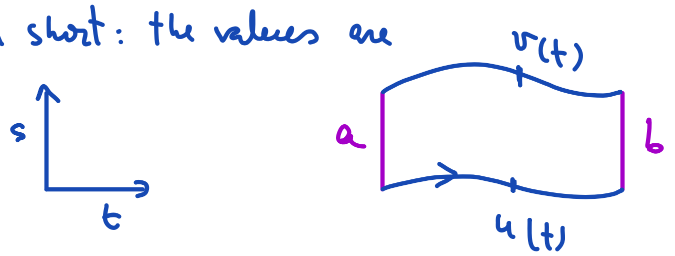
Definition: Fix X top space, $a, b \in X$ and two curves $u, v: [0,1] \longrightarrow X$ with $u(0) = v(0) = a$ & $u(1) = v(1) = b$. A <u>homotopy</u> between u & v is a continuous function $H: [0,1] \times [0,1] \longrightarrow X$ with

(i) $H(t,0) = u(t) \quad \forall\, t \in [0,1]$

(ii) $H(t,1) = v(t) \quad \forall\, t \in [0,1]$

(iii) $H(0,s) = a$ & $H(1,s) = b \quad \forall\, s \in [0,1]$

In short: the values are

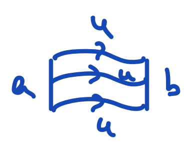


By construction for each $s \in [0,1]$ we get a curve $u_s(t) = H(t,s)$. The family of curves $\{u_s(t)\}_{0 \le s \le 1}$ is called a deformation (or homotopy) of u into v.

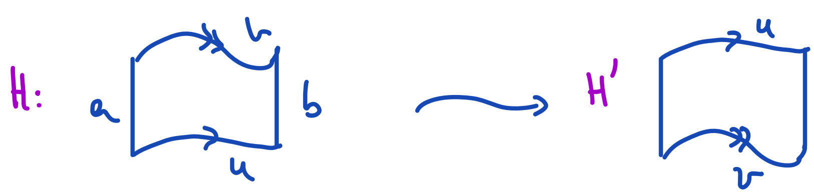
Definition: Two curves u, v are <u>homotopic</u> if $u(0) = v(0)$ & $u(1) = v(1)$ & $\exists\, H: [0,1] \times [0,1] \longrightarrow X$ homotopy between u & v. We write $u \sim v$.

Lemma 1: Homotopy determines an equivalence relation in the space of paths with fixed starting and ending points.

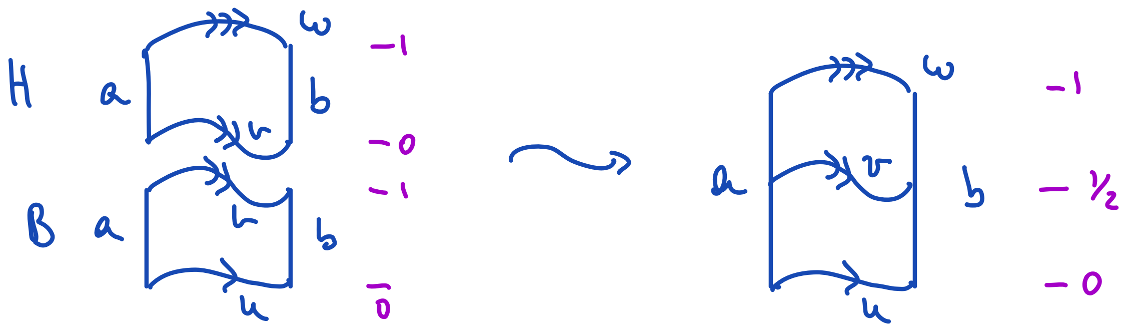
Pf/. • $u \sim u$ via $H(t,s) = u(t) \quad \forall\, s \in [0,1]$



• If $u \sim v$ via H, then $v \sim u$ via $H'(t,s) = H(t, 1-s) \;\forall s$.



• If $u \sim v$ via H & $v \sim w$ via B, then $u \sim w$ via $C: [0,1] \times [0,1] \longrightarrow X$ with $C(t,s) = \begin{cases} H(t, 2s) & \text{if } 0 \le s \le 1/2 \\ B(t, 2s-1) & \text{if } 1/2 \le s \le 1 \end{cases}$



C is continuous by construction. $\square$

Lemma 2: Fix a curve $u:[0,1] \to X$ & $\varphi:[0,1] \to [0,1]$ is continuous, with $\varphi(0)=0$, $\varphi(1)=1$ then $u \sim u \circ \varphi$.

Proof By construction $u \circ \varphi : [0,1] \to X$ is a curve & $u\circ\varphi(0) = u(0)$
$u \circ \varphi(1) = u(1)$

Define $A : [0,1] \times [0,1] \longrightarrow X$ via $A(t,s) = u((1-s)t + s\varphi(t))$

Then, A is continuous and
- $A(t,0) = u(t)$
- $A(t,1) = u\circ\varphi(t)$ $\quad \forall s,t$
- $A(0,s) = u(s\varphi(0)) = u(0)$
  $A(1,s) = u((1-s) + s\cdot 1) = u(1)$

So A is the desired homotopy.

Lemma 3: Concatenation, opposite operation factor through homotopy equiv.

More precisely, if $u_1 \sim u_2$ & $v_1 \sim v_2$ with $u_1(1) = v_1(0)$, then

(1) $u_1 \bullet v_1 \sim u_2 \bullet v_2$

(2) $u_1^{op} \sim u_2^{op}$

Proof: Exercise

Lemma 4: If $u:[0,1] \to X$ is a curve with $a = u(0)$, $b = u(1)$

then $u \bullet u^{op} \sim \mathbb{1}_a$ & $u^{op} \bullet u \sim \mathbb{1}_b$.

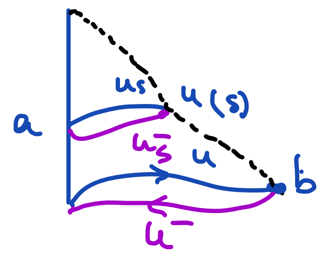
Proof: By symmetry, it's enough to build a homotopy $H:[0,1]\times[0,1]\to X$ between $u \cdot u^{op}$ & 1_a

Recall: $u \cdot u^{op}(t) = \begin{cases} u(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ u^{op}(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$

Notice: $u^{op}(2t-1) = u(1-(2t-1)) = u(2-2t)$

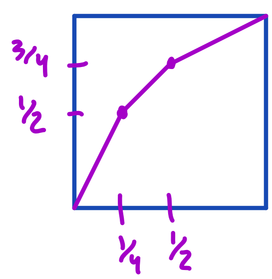
Now, define $H:[0,1]\times[0,1] \longrightarrow X$ by $H(t,s) = \begin{cases} u(2t(1-s)) & \text{if } 0 \le t \le \frac{1}{2} \\ u(2(1-t)(1-s)) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$

(Idea: keep stopping half-way through u & concatenate with the corresponding reverse curve)



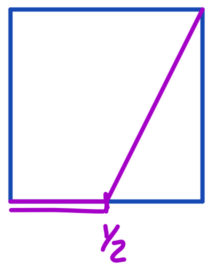
Lemma 5: $(u \cdot v) \cdot w \sim u \cdot (v \cdot w)$ whenever the curve can be concatenated.

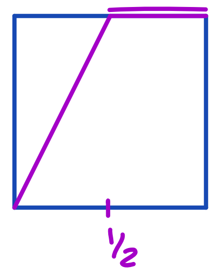
Proof: It is easy to see that $(u \cdot v) \cdot w = (u \cdot (v \cdot w)) \circ \varphi$ where $\varphi:[0,1] \longrightarrow [0,1]$ is the piecewise affine linear function defined by



Then $u \cdot (v \cdot w) \circ \varphi \sim u \cdot (v \cdot w)$ by Lemma 2.

Lemma 6: If u is a curve in X, then $1_{u(0)} \cdot u \sim u \sim u \cdot 1_{u(1)}$.

Proof $1_{u(0)} \cdot u = u \circ \varphi$ when φ is the piecewise affine linear function given by  Since $u \circ \varphi \sim u$ by Lemma 3, the result holds.

Similarly, $u \circ 1_{u(1)} = u \circ \psi$ where ψ is given by 
so $u \circ 1_{u(1)} \sim u$ by Lemma 3.

§3.2. Path connected, Locally path connected & connected spaces

Def: • We say X is pathwise (or arcwise) connected if $\forall x, y \in X$ we have a curve $u : [0,1] \longrightarrow X$ with $u(0) = x$ & $u(1) = y$

• We say X is locally pathwise connected if $\forall x \in X$ $\exists$ a neighborhood basis $\mathcal{B}(x)$ of pathwise connected open sets containing x (This means that for any neighborhood U of x $\exists$ V open & pathwise connected with $x \in V \subseteq U$)

• We say X is connected if, and only if $\emptyset$ & X are the only subsets of X that are both open & closed.

Lemma 1: Pathwise connected sets are connected

Pf/ We argue by contradiction & write $X = X_1 \sqcup X_2$ with both X_1 & X_2 open & non-empty. Pick $x_1 \in X_1 \setminus X_2$ & $x_2 \in X_2 \setminus X_1$.

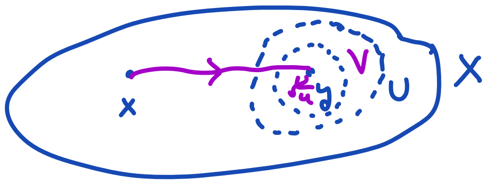
Since X is globally path connected, then $\exists$ $u : [0,1] \longrightarrow X$ cont. with $u(0) = x_1$ & $u(1) = x_2$.

Since u is continuous $\underset{0 \in}{u^{-1}(X_1)}$, $\underset{1 \in}{u^{-1}(X_2)}$ are both open in $[0,1]$, non-empty and disjoint sets covering $[0,1]$. Thus, $[0,1] = u^{-1}(X_1) \sqcup u^{-1}(X_2)$, which cannot happen because $[0,1]$ is connected.

Lemma 2: A connected & locally pathwise connected space is pathwise connected.

Pf/ Fix $x \in X$ & $H = \{y \in X : x \,\&\, y$ can be joined by a curve in $X\}$

- Since X is locally path connected, H is open



Fix $y \in H$ & $y \in U \subseteq X$ open.
We have $y \in V \underset{\text{open}}{\subseteq} U$ & V path connected
any curve joining y to a point $v \in V$ can be concatenated to a curve joining x & u.
(see §3.1)

- Similarly, we can show that $X \setminus H$ is open.

But $x \in H$ & X is connected, so $H = X$ & thus X is (globally) path connected. $\square$

⚠ $\exists$ spaces that are pathwise connected but NOT locally pathwise connected

Example $X =$

No neighborhood U of $(0,1)$ has a pathwise connected open V with $(0,1) \in V \subseteq U$

§3.3 Loops, Fundamental groups

Definition: A *loop* on X is a curve $u : [0,1] \to X$ with $u(0) = u(1)$.

Definition: A loop u is *null-homotopic* if $u \sim 1_{u(0)}$ (homotopy equivalent)

Definition: Given $a \in X$, we define $\pi_1(X,a)$ as the set of equivalence classes of loops based at a modulo homotopy.

Theorem: For each $a \in X$, the set $\pi_1(X,a)$ is a group under concatenation, with neutral element 1_a & inverse operation given by reversing a curve ($u \mapsto \bar{u}$)

If X is path connected, we call $\pi_1(X,a)$ the *fundamental group* of X

Furthermore, it is independent of the chosen base point a. More precisely, if $u:[0,1]\to X$ is a curve with $u(0)=a$ & $u(1)=b$, then $\pi_1(X,a)\simeq\pi_1(X,b)$ via

$$\begin{array}{ccc} \pi_1(X,a) & \xrightarrow{\Psi} & \pi_1(X,b) \\ \gamma & & \bar{u}\bullet\gamma\bullet u \end{array}$$

Proof: • Lemma 3 in §4.1 ensure concatenation & inverse operations are compatible with the equivalence relation $\sim$. Associativity holds by Lemma 5; $\mathbb{1}_a$ is the neutral element by Lemma 6. Lemma 4 ensures the inverse path operation is the inverse in $\pi_1(X,a)$.

• Ψ is well-defined by Lemmas 3 & 5 from §4.1. Ψ is invertible via

$$\begin{array}{ccc} \pi_1(X,b) & \longrightarrow & \pi_1(X,a) \\ \alpha & \longmapsto & u\bullet\alpha\bullet\bar{u} \end{array}$$

$\square$

Definition: We say X is simply connected if X is path connected & $\pi_1(X,a)=\{\mathbb{1}_a\}$ (the fundamental group is trivial, usually denoted additively by 0).

Proposition: If X is simply connected, any 2 curves with the same starting & ending points are homotopic.

Proof: Pick $u,v:[0,1]\to X$ curves with $u(0)=v(0)=a$ & $u(1)=v(1)=b$.

Then $u\bullet\bar{v}\in\pi_1(X,a)=\{\mathbb{1}_a\}$ so $u\bullet\bar{v}\sim\mathbb{1}_a$.

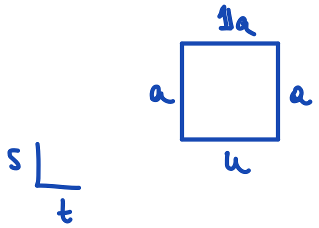
Thus $(u\bullet\bar{v})\bullet v\sim u\bullet(\bar{v}\bullet v)\sim u\bullet\mathbb{1}_a\sim u$, so u & v are homotopic

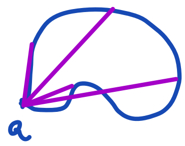
$$\begin{array}{c} (u\bullet\bar{v})\bullet v \\ \| \\ \mathbb{1}_a\bullet v\sim v \end{array}$$

Examples: (1) Star-shaped subsets of $\mathbb{R}^n$ (ie $\exists\, a\in X$ s.t. $\forall x\in X$ the segment $[a,x]=\{\lambda a+(1-\lambda)x : 0\leq\lambda\leq1\}$ lies in X) are simply connected.

Why? By construction it's path connected. Given $u:[0,1]\to X$ loop with base a, we have a homotopy between u & $\mathbb{1}_a$ via:

$$H : [0,1] \times [0,1] \longrightarrow X \qquad H(t,s) = s\cdot a + (1-s)\,u(t)$$





(move along segments)

(2) $\mathbb{C}$ & every disc in $\mathbb{C}$ is simply connected (they are star-shaped)

Proposition 2: Pick $U_1, U_2 \subseteq X$ open with $X = U_1 \cup U_2$. Assume U_1, U_2 are open, simply connected subsets & $U_1 \cap U_2$ is path connected. Then, X is simply connected.

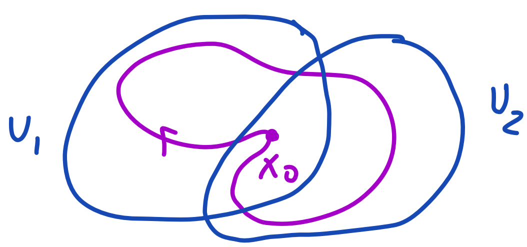
Proof (Proposition 3.2.1 in Aguilar, Gitler & Prieto "Algebraic Topology from a Homotopical viewpoint")

Pick $x_0 \in U_1 \cap U_2$ & a loop $u : [0,1] \longrightarrow X$ based at x_0.

- If $\mathrm{im}(u) \subseteq U_1$ or $\mathrm{im}(u) \subseteq U_2$, we conclude $u \sim \mathbb{1}_{x_0}$ in $\pi_1(U_1, x_0)$ or $u \sim \mathbb{1}_{x_0}$ in $\pi_1(U_2, x_0)$ so u is null-homotopic in X.
- On the contrary, assume $\mathrm{im}(u) \not\subseteq U_1$ & $\mathrm{im}(u) \not\subseteq U_2$. By construction, $\{u^{-1}(U_1), u^{-1}(U_2)\}$ is an open cover of $[0,1]$. It is non-trivial by construction.
- There exists a number $\delta > 0$ (called the Lebesgue number of this cover) such that if $0 \leq t - s \leq \delta$, then $[s,t] \subseteq u^{-1}(U_1)$, or $[s,t] \subseteq u^{-1}(U_2)$. [Hint: Use $[0,1]$ is compact]

Hence, we can partition $[0,1]$ by $0 = t_0 < t_1 < \cdots < t_k = 1$ so that

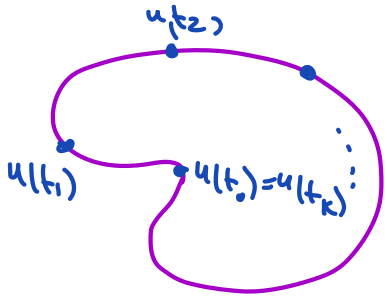
$$u[t_0,t_1] \subseteq U_1, \quad u[t_1,t_2] \subseteq U_2, \quad u[t_2,t_3] \subseteq U_3, \quad \ldots, \quad u[t_{k-1},t_k] \subseteq U_2$$



In particular, $u(t_i) \in U_1 \cap U_2$ $\forall i = 1, \ldots, k-1$.

- We can realize u as a concatenation of paths via homotopy:

$$u \sim u_0 \cdot u_1 \cdot \dots \cdot u_{k-1} \quad \text{with} \quad u_i = u|_{[t_i, t_{i+1}]}.$$

- 

Since $u(t_i) \in U_1 \cap U_2 \quad \forall\, i = 1, \dots, k-1$

we can find a curve $\gamma_i : [0,1] \longrightarrow U_1 \cap U_2$ joining x_0 & $u(t_i)$.

Set $\gamma_0 = 1\!\!1_{w_0}$ & $\gamma_k = 1\!\!1_{w_0}$.

- Then: $\gamma_i \cdot u|_{[t_i, t_{i+1}]} \cdot \gamma_{i+1}^-$ is a loop in either U_1 or U_2 based at w_0

Since both $\pi_1(U_1, x_0)$ & $\pi_1(U_2, x_0)$ are trivial, we get $\gamma_i \cdot u|_{[t_i, t_{i+1}]} \cdot \gamma_{i+1}^- \sim 1\!\!1_{w_0}$

In addition $1\!\!1_{u(t_i)} \sim \gamma_i^- \cdot \gamma_{i+1}$ since both γ_i & γ_{i+1} are curves in $U_1 \cap U_2$.

& the homotopy can be performed in either U_1 or U_2.

Thus,

$$\begin{aligned}
u &\sim (\gamma_0 \cdot u_0 \cdot 1\!\!1_{u(t_1)} \cdot u_1 \cdot 1\!\!1_{u(t_2)} \cdot u_2 \cdot \dots \cdot 1\!\!1_{u(t_{k-1})} \cdot u_{k-1} \cdot \gamma_k \\
&\overset{\text{Assoc}}{\sim} (\gamma_0 \cdot u_0 \cdot \gamma_0^-) \cdot (\gamma_1 \cdot u_1 \cdot \gamma_1^-) \cdot (\gamma_2 \cdot \dots \cdot u_{k-2} \cdot \gamma_{k-1}^-) \cdot (\gamma_{k-1} \cdot u_{k-1} \cdot \gamma_k) \\
&\sim \underbrace{1\!\!1_{w_0} \cdot 1\!\!1_{w_0} \cdot 1\!\!1_{w_0} \cdot \dots \cdot 1\!\!1_{w_0}}_{k+1 \text{ times}} = 1\!\!1_{w_0}.
\end{aligned}$$

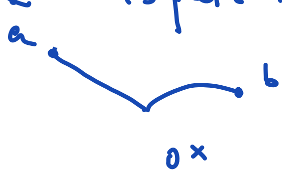
So $\pi_1(X, x_0) = \{0\}$. $\square$

Corollary: $\mathbb{P}^1$ is simply connected

Pf/ Cover $\mathbb{P}^1$ with $\left(U_0 = \mathbb{P}^1 \setminus \{\infty\},\ \varphi_0 : \mathbb{P}^1 \setminus \{\infty\} \xrightarrow{\sim} \mathbb{C}\right)$ &
$\qquad z \longmapsto z$
$\left(U_\infty = \mathbb{P}^1 \setminus \{0\},\ \varphi_\infty : \mathbb{P}^1 \setminus \{\infty\} \xrightarrow{\sim} \mathbb{C}\right)$
$\qquad z \longmapsto 1/z$

- $U_0 \underset{\text{homeo}}{\simeq} \mathbb{C}$ & $U_\infty \underset{\text{homeo}}{\simeq} \mathbb{C}$ are simply connected by Example (2) §3.3
- $U_0 \cap U_\infty = \mathbb{C}^\times$ is path connected via circular arcs & segments through 0.

Eg:



By Proposition 2 §3.3 $\mathbb{P}^1$ is simply connected. $\square$

Proposition: (Functorial Behavior)

Fix X, Y top spaces & $f: X \longrightarrow Y$ continuous. If $u: [0,1] \longrightarrow X$ is a curve, then $f \circ u : [0,1] \longrightarrow Y$ is a curve. Moreover, if $u \sim v$ as curves in X, then $f \circ u \sim f \circ v$ as curves in Y (Take $f \circ H$ where H is the homotopy between u & v). Thus, we get a map:

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$$

This map is a group homomorphism since $f \circ (u \bullet v) = (f \circ u) \bullet (f \circ v)$

Furthermore if $g: Y \longrightarrow Z$ is continuous, then $(g \circ f)_* = g_* \circ f_*$.

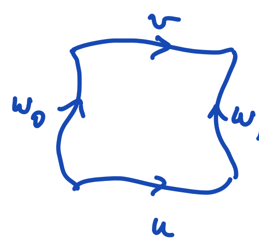
§4.4 Free-homotopies:

Fix a Top space X & curves $u, v: [0,1] \longrightarrow X$ which do not necessarily have the same initial point.

Def: We say u & v are free homotopic if there exists a continuous map $A: [0,1] \times [0,1] \longrightarrow X$ with

(1) $A(t,0) = u(t) \quad \forall t \in [0,1]$

(2) $A(t,1) = v(t) \quad \forall t \in [0,1]$



Remarks: (1) $u_s(t) = A(t,s)$ is a curve $\forall s \in [0,1]$.

Then $w_0(s) = A(0,s) : [0,1] \longrightarrow X$
$w_1(s) = A(1,s) : [0,1] \longrightarrow X$ defines a curve joining $u(0)$ & $v(0)$, $u(1)$ & $v(1)$

Lemma: $u \sim w_0 \bullet v \bullet w_1^-$

(2) If u & v are loops, then we ALSO require (3) $A(0,s) = A(1,s) \quad \forall s \in [0,1]$
$\Rightarrow u_s$ is a loop based at $A(0,s)$ $\forall s \in [0,1]$ & $w_0 = w_1$

(3) In the loop setting, $w_0(s)$ is the base point of the loop u_s.