Lecture IV: Branch Pts, liftings relative to local homeomorphism

Recall: Last Time we discussed (local) behavior of holomorphic functions between RS

- **Open Mapping Thm:** $f: X \longrightarrow Y$ non constant holomorphic map between R.S. is open
- **Identity Theorem:** If $f, g: X \to Y$ holomorphic maps agree on an infinite set with a limit point, then $f = g$
- **Local Behavior:** $f: X \longrightarrow Y$ non-const. holomorphic map between R.S, $a \in X$ & $b = f(a) \in Y$

Then $\exists$ N & coordinate charts $\left(\begin{array}{l} U \xrightarrow[\sim]{\varphi_U} \mathbb{D}' \\ V \xrightarrow[\sim]{\varphi_V} \mathbb{D} \end{array}\right)$ with $a \in U \underset{\text{open}}{\subseteq} X$ & $b \in V \underset{\text{open}}{\subseteq} Y$ st

(1) $\varphi_U(a) = 0$, $\varphi_V(b) = 0$, (2) $f(U) \subseteq V$

(3) Locally: $\varphi_V \circ f \circ \varphi_U^{-1}: \mathbb{D} \longrightarrow \mathbb{D}$ is $z \longmapsto z^N$

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "z"'] & V \arrow[d, "z"] \\ \mathbb{D} \arrow[r] & \mathbb{D}' \end{tikzcd} \qquad z \longmapsto z^N$$

Equivalently: $\forall$ U_0 nbhd of a $\exists$ $W \subseteq Y$ nbhd of b & $a \subseteq U \subseteq U_0$ open st $f^{-1}(y) \cap U$ has N elements $\forall$ $y \neq b$. (coord. free characterization of N)

Name: N = branching number of f near a = multiplicity of f at a.

§4.1 Branch points:

Fix $f: X \longrightarrow Y$ non-constant holomorphic map between R.S.

Proposition: For each $y \in Y$, the fiber $f^{-1}(y)$ is discrete

Pf/ If not, by the Identity Theorem, f is constant. ($\equiv y$). $\square$

Def: A point $x_0 \in X$ is a branch point (or ramification pt) of f if there is no open set $U \subseteq X$ with $x_0 \in U$ s.t $f|_U$ is injective

Def: We say f is unbranched (or unramified) if it is non-constant & has no branch pts.

Consequence 1: A = set of branch points is discrete & closed

Pf/ $X \setminus A$ is open ($z_0 \notin A \Rightarrow f|_{U \ni z_0}$ is injective for some $z_0 \in U \underset{\text{open}}{\subseteq} X$).

Examples: (1) $f: \mathbb{C} \longrightarrow \mathbb{C}$, $z \longmapsto z^N$ for $N > 1$, 0 is the only branch pt

(2) $f|_{\mathbb{C}^*} : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ $f(z) = z^N$ is unbranched.

(3) $\exp: \mathbb{C} \longrightarrow \mathbb{C}^*$ is unbranched $p^{-1}(x) = y + 2\pi i \mathbb{Z}$ for $y = \log(x)$ (local branch of log near x)

(4) $\mathbb{C} \xrightarrow{\pi} \mathbb{C}/\Gamma$ for $\Gamma = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ discrete rank 2 lattice is unbranched

Theorem 1: Fix $f: X \longrightarrow Y$ non-constant holomorphic function. TFAE:

(1) f is unbranched (locally injective)

(2) f is a local homeomorphism (ie $\forall\, x \in X$ $\exists\, U \subseteq X$ open with $x \in U$ & $V \subseteq Y$ open with $f(x) \in V$ & $f|_U : U \longrightarrow V$ is homeomorphism)

(3) f is a local biholomorphism

Proof: (1) $\Rightarrow$ (2): f has branching number $= 1$ so f is open & locally injective $\Rightarrow$ local homeomorphism (f is open & locally injective $\Rightarrow$ locally homeo)

(2) $\Rightarrow$ (3) Local homeomorphisms are non-constant & locally injective, so $f|_U : U \longrightarrow V = f(U)$ is bijective & holomorphic. By HW1 Problem 8, $f|_U$ is biholomorphic. (Alternatively: Use Inverse Function Thm (Corollary 2, § 2.4))

(3) $\Rightarrow$ (1) Local biholomorphism are locally injective, hence unbranched. $\square$

Theorem 2: Fix Y a RS, X a Hausdorff connected Top space and $p: X \longrightarrow Y$ a local homeomorphism. Then, there is a unique complex structure on X making p holomorphic.

Proof: Idea: Pull back the complex structure from Y to X via the local homeomorphism p

- $\mathbb{C}$-charts: Pick $x_0 \in X$. Since p is a local homeo, $\exists$ opens U_0, V_0 with

$x_0 \in U_0 \underset{\text{open}}{\subseteq} X$ & $y_0 = p(x_0) \in V_0 \underset{\text{open}}{\subseteq} Y$ st $p|_{U_0} : U_0 \longrightarrow V_0$ is a homeomorphism

Pick $y_0 \in V \subseteq V_0$ coordinate chart with $\varphi: V \xrightarrow{\sim} \mathbb{D} \subseteq \mathbb{C}$ homeo. Then

$U = p_{|U_0}^{-1}(V) = p^{-1}(V) \cap U_0$ satisfies

(1) $x_0 \in U$

(2) $U \subseteq X$ is open

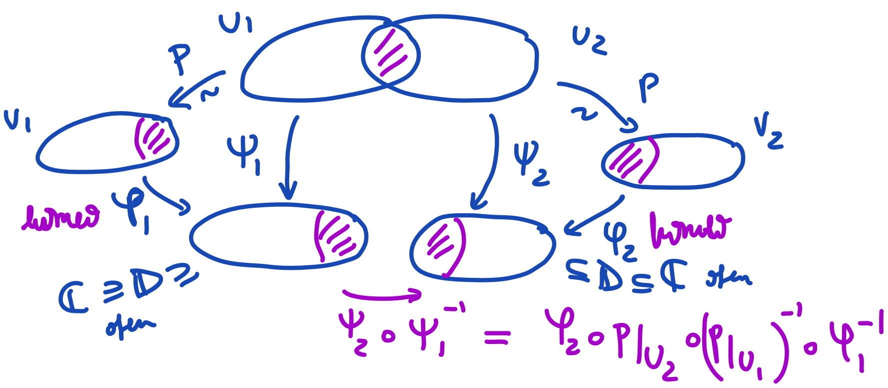
(3) $\psi: \varphi \circ p_{|U} : U \xrightarrow{\sim} V \xrightarrow{\sim} \mathbb{D}$ is homeo

- Compatibility: We need to check if (U_1, ψ_1), (U_2, ψ_2) are coord charts ($x_1 \in$, $x_2 \in$) then $\psi_1 = \varphi_1 \circ p|_{U_1}$ & $\psi_2 = \varphi_2 \circ p|_{U_2}$ with $p_{|U_i} : U_i \xrightarrow{\sim} V_i$ homeo

Pick $x_0 \in U_1 \cap U_2$. Then $\psi_2 \circ \psi_1^{-1} : \psi_1(U_1 \cap U_2) \longrightarrow \psi_2(U_1 \cap U_2)$ equals $\varphi_2 \circ \varphi_1^{-1}$

$\psi_1(U_1 \cap U_2) = \varphi_1(p(U_1 \cap U_2))$, $\psi_2(U_1 \cap U_2) = \varphi_2(p(U_1 \cap U_2))$

which is a biholomorphism between opens in $\mathbb{C}$



- p is holomorphic: Pick $x_0 \in U \subseteq X$ open chart with $p|_U : U \longrightarrow p(U) = V$ homeo $(U, \psi = \varphi \circ p : U \xrightarrow{\sim} \mathbb{D} \subseteq \mathbb{C})$ Then

$$\begin{array}{ccc} x_0 \in U & \xrightarrow{p} & V \ni p(x_0) \\ \psi \downarrow & & \downarrow \varphi \\ \mathbb{D} & \xrightarrow[=]{} & \mathbb{D} \end{array}$$

& $id_{\mathbb{D}}$ is holomorphic.

- Uniqueness: Assume X admits two complex structures Σ & Σ_1 making p into a holomorphic map. Consider the diagram

$$\begin{array}{ccc} (X,\Sigma) & \xrightarrow{id} & (X,\Sigma_1) \\ & p \searrow \;\circlearrowleft\; \swarrow p & \\ & Y & \end{array}$$

- p is holomorphic & local homeo $\Rightarrow$ locally biholomorphic by Thm 1 §4.1

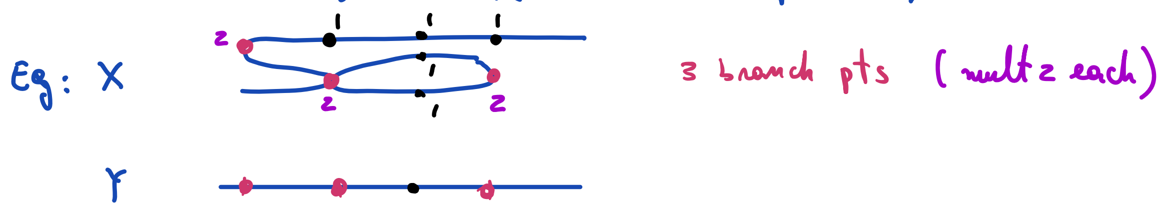
$\Rightarrow$ id is a local biholomorphism, hence a biholomorphism. Thus $(X,\Sigma) \underset{id}{\simeq} (X,\Sigma_1)$ shows $\Sigma = \Sigma_1$.

Next Goal: Count points in fibers of non-constant holomorphic maps
$f: X \to Y$ (with multiplicities) & show it's the same for all fibers.

Idea: If f is locally injective at $x_0 \in f^{-1}(y)$, we count it with mult 1.

If x_0 is a branch pt, then we count x_0 with mult = branching number of f at x_0

We'll see that if X is compact (or f is proper), then all fibers are finite & have the sizes (counted with multiplicities)

Eg: 

$\rightsquigarrow$ We get a notion of degree for proper holomorphic non-constant maps.
(Next Time!) ↳ preimages of compact sets are compact.

Ex: $X \xrightarrow{f} \mathbb{P}^1$ holo, non constant & X compact, then

$$\deg(f) = \begin{cases} |f^{-1}(0)| = \#\text{ of zeroes (counted with mult)} \\ |f^{-1}(\infty)| = \#\text{ of poles (counted with mult)} \end{cases}$$

- To define $\deg f$ we'll need an interlude on covering maps between top spaces & lifting cont maps through local homeomorphisms. (TODAY & Next Time!)

- These topological tools will be used to build new Riemann surfaces as quotients of universal covers of R.S.

§4.2 Lifting (or Factoring) maps via local homeomorphisms:

General Setting: X, Y, Z topological spaces &

$$\begin{tikzcd} & X \arrow[d, "p\ \text{local homeo}"] \\ Z \arrow[r, "f"', "\text{cont}"] & Y \end{tikzcd}$$

with $z_0 \in Z$ (& $x_0 \in X$ st $p(x_0) = f(z_0)$.)

Q1: Can we factor f through X (with value x_0 at z_0) ?

$$\begin{tikzcd} & X \ni x_0 \arrow[d, "p\ \text{local homeo}"] \\ z_0 \in Z \arrow[ur, dashed, "?\exists \hat{f}"] \arrow[r, "f"', "\text{cont}"] \arrow[ur, phantom, shift right=3ex, "\circlearrowleft"] & Y \ni f(z_0) = p(x_0) \end{tikzcd}$$

with
- $\hat{f}$ cont
- $p \circ \hat{f} = f$
- ($\hat{f}(z_0) = x_0$) $\rightsquigarrow$ "initial condition"

Def: If $\hat{f}$ exists, we call it a <u>lifting of f with respect to p</u>.

Q2: Are liftings unique?

A: Summary of Results:

① Local liftings always exist & will be unique under connectedness assumptions on Z + Hausdorff assumptions on X & Y.

② If curves $u: [0,1] \longrightarrow Y$ can be lifted wrt p & Z is nice (simply connected & locally path connected) then any $f: Z \to X$ cont. can be lifted w.r.t. p

③ This "curve lifting property" holds for (topological) covering.

Def: A cont map $p: X \longrightarrow Y$ is a <u>covering</u> if $\forall y \in Y$ $\exists$ $V \subseteq Y$ open with $y \in V$ & a collection $\{U_j\}_{j \in J}$ of pairwise disjoint opens of X with

(1) $p^{-1}(V) = \bigsqcup_{j \in J} U_j$

(2) $p|_{U_j} : U_j \longrightarrow V$ is a homeomorphism

Lemma 1: Covering maps are local homeomorphisms

Pf/ Pick $x \in X$ & $p(x) = y$ Then $\exists$ $y \in V \subseteq Y$ open & $\{U_j\}$ as in the definition. By (1), $\exists!\, j$ with $x \in U_j$. By (2), $p|_{U_j} : U_j \xrightarrow{\sim} V_j$ homeo.

§4.3. Local liftings w.r.t. local homeomorphisms / Uniqueness:

Setting: X, Y Hausdorff Top spaces & $p: X \to Y$ local homeomorphism

Theorem 1: Given $f: Z \to X$ & $z_0 \in Z$, $x_0 \in X_0$ with $p(x_0) = f(z_0)$,

$\exists \underset{\text{open}}{W \subseteq Z}$ with $x_0 \in W$ & $\hat{f}: W \to X$ cont. lifting of $f_{|_W}$ w.r.t. p with $\hat{f}(z_0) = x_0$

Furthermore, $\hat{f}$ is unique (Two local lifts will agree on W' open with $z_0 \in W' \subseteq W_1 \cap W_2$)
(as a germ at z_0)

Proof: Given $x_0 \in X$ & $y_0 = p(x_0) \in Y$, pick U, V opens with $x_0 \in U \subseteq X$, $y_0 \in V \subseteq Y$
& $p_{|_U}: U \to V$ homeo. Take $W = f^{-1}(U)$. Then

- $z_0 \in W$
- W is open (f is continuous)
- $\hat{f} = p_{|_U}^{-1} \circ f_{|_W} : W \to X$ is cont. & $p \circ \hat{f} = f$

. $\hat{f}$ is unique because p is a local homeomorphism: any other lift has this formula after restricting its domain to a smaller neighborhood of z_0. □

Theorem 2: (Uniqueness of Global Liftings)

Assume Z is connected & let $g_1, g_2 : Z \to X$ be two (global) liftings of f relative to p. If $g_1(z_0) = g_2(z_0)$ for some $z_0 \in Z$, then $g_1 = g_2$.

Pf/ Let $T = \{z \in Z : g_1(z) = g_2(z)\}$

- $T \neq \emptyset$ since $z_0 \in T$
- T is closed since $T = (\underset{\text{cont.}}{g_1 \times g_2})^{-1}(\Delta)$ where $\Delta = \{(y,y) : y \in Y\}$ is the diagonal (closed set).
- T is open since f admits ! local lifts around z_0 with value $g_1(z_0)$ at z_0 by Theorem 1. ($\exists$ $W \subseteq Z$ open with $z_0 \in W$ & $\hat{f} = g_1 = g_2$ on W).

Since Z is connected, we have $T = Z$ so $g_1 \equiv g_2$. □

Theorem 3 (Liftings for R.S. & holomorphic maps)

Assume X, Y are R.S & $p: X \to Y$ is unbranched & holomorphic
(⇒ local homeo)

If Z is a RS & $f: Z \to Y$ is holomorphic, then every lifting $\hat{f}: Z \to X$ of f relative to p is holomorphic.

Pf/ Unbranched holomorphic maps are locally biholomorphic by Thm 1 §4.1
Pick $z_0 \in Z$ & $x_0 = \hat{f}(z_0)$. Then by Thm 2 §4.3, $\hat{f}$ agrees with $(p_{|U})^{-1} \circ f$, where $p_{|U}: U \to V$ is biholo & $x_0 \in U$ & $y_0 = p(x_0) \in V$

Since f & $(p_{|U})^{-1}$ are holo, so is $\hat{f}$. □

§4.4 Lifting curves & homotopies:

Setting: X, Y Hausdorff Top spaces & $p: X \to Y$ local homeomorphism

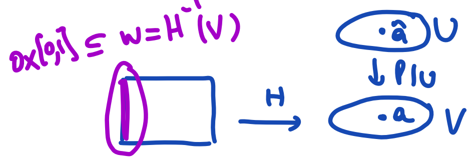
Theorem 1: (Lifting homotopic curves) Assume $a, b \in Y$ & $\hat{a} \in X$ with $p(\hat{a}) = a$
Assume $H: [0,1] \times [0,1] \to Y$ is continuous with $H(0,s) = a$ & $H(1,s) = b$ $\forall s \in [0,1]$ Consider the collection of paths $u_s(t) = H(t,s)$ joining a & b in Y.

If every path u_s lifts to a curve $\hat{u}_s$ relative to p with $\hat{u}_s(0) = \hat{a}$ then
$$\hat{H}: [0,1] \times [0,1] \to X \quad \text{with} \quad \hat{H}(t,s) = \hat{u}_s(t)$$
is a continuous lifting of H relative to p. Furthermore, $\hat{H}$ is a homotopy between $\hat{u}_0$ & $\hat{u}_1$. In particular $\hat{u}_0$ & $\hat{u}_1$ have the same end point.

Proof: We need to show continuity & $\hat{H}(1,\_)$ is constant. If so, the construction yields $p \circ \hat{H}(t,s) = p \circ \hat{u}_s(t) = u_s(t) = H(t,s)$ & $\hat{H}(0,s) = \hat{a}$ so $\hat{u}_0 \sim \hat{u}_1$ via $\hat{H}$

We show continuity by working over a grid in $[0,1]\times[0,1]$

Fix $\hat{a} \in \underset{\text{open}}{U} \subseteq X$ & $a \in \underset{\text{open}}{V} \subseteq Y$ with $p|_U : U \to V$ local homeo



H is continuous & $H(0,s) = a \in V$.

Thus, $W := H^{-1}(V) \subseteq [0,1]\times[0,1]$ is open & $\{0\}\times[0,1] \subseteq W$

• <u>Claim 1</u>: $\exists\ \varepsilon_0 > 0$ s.t. $[0,\varepsilon_0)\times[0,1] \subseteq W$ & $\hat{H}|_{[0,\varepsilon_0)\times[0,1]}$ is continuous

Pf/ For each $s \in [0,1]$ $\exists\ \underset{\alpha_s < s}{\delta_s} > 0$ with $W_s := [0,\delta_s) \times [s-\alpha_s, s+\alpha_s]$.

The collection $\{W_s\}_{s\in[0,1]}$ covers the compact set $\{0\}\times[0,1]$ so we have a finite subcover $\{W_{s_1}, \dots W_{s_n}\}$. Take $\varepsilon_0 = \min\{\delta_1, \dots, \delta_n\} > 0$ &

$[0,\varepsilon_0)\times[0,1] \subseteq W$.

By uniqueness of liftings from connected Z, since $\hat{u}_s(0) = \hat{a}$ $\forall s$ we get $\hat{u}_s = (p|_U)^{-1} \circ u_s$ by Theorem 1 §4.3. Thus, we get $\hat{H}|_{[0,\varepsilon_0)\times[0,1]} = (p|_U)^{-1} \circ H|_{[0,\varepsilon_0)\times[0,1]}$. Hence, the restriction $\hat{H}|_{[0,\varepsilon_0)\times[0,1]}$ is continuous. $\square$

• <u>Claim 2</u>: $\hat{H}$ is continuous.

Pf/ We argue by contradiction. Assume $\exists\,(t_0,s_0)$ with $\hat{H}$ disc. at (t_0,s_0).

Write $\tau = \inf\{t \,/\, \hat{H} \text{ is not cont at } (t, s_0)\}$

By Claim 1 $\tau \geq \varepsilon_0$.

Write $\hat{c} = \hat{H}(\tau, s_0)$ & $c = H(\tau, s_0)$, so $p(\hat{c}) = c$. Since p is local homeomorphism, pick $\hat{c} \in U' \subseteq X$ & $c \in V' \subseteq Y$ with $p|_{U'} : U' \to V$ local homeo. Pick $W' := H^{-1}(V') \subseteq Z$ open

• <u>Note</u>: $(\tau, s_0) \in W'$ & $\exists\ \underset{\delta>0}{\varepsilon > 0}$ with $\underbrace{(\tau-\varepsilon, \tau+\varepsilon)\times(s_0-\delta, s_0+\delta)}_{\text{connected}} \subseteq W'$

By construction (Uniqueness of Local Lifts) $\hat{H}\big|_{(\tau-\varepsilon,\tau+\varepsilon)\times(s_0-\delta,s_0+\delta)} = (p|_{U'})^{-1}\circ H\big|_{(\tau-\varepsilon,\tau+\varepsilon)\times(s_0-\delta,s_0+\delta)}$

so $\hat{H}$ is continuous on $(\tau,\tau+\varepsilon)\times\{s_0\}$, contradicting the def of τ.

<u>Claim 3:</u> $\hat{H}(1,s) = \hat{u}_0(1)$ $\forall s$.

Pf/ H is a homotopy, so $H(1,s) = u_0(1) = b$ $\forall s$.

Since p is local homeo, then $p^{-1}(b)$ is discrete.

Now: $\hat{H}(1,-): [0,1] \longrightarrow Z$ is a lifting of $H(1,-)\equiv b$.

with $\hat{H}(1,0) = \hat{u}_0(1)$, so $\hat{H}(1,s) \subseteq p^{-1}(b)$ $\forall s\in[0,1]$

But $\hat{H}(\{1\}\times[0,1])$ is connected in a discrete set, so it is constant

$\Rightarrow \hat{H}(1,s) = \hat{H}(1,0) = \hat{u}_0(1)$ $\forall s$ in $[0,1]$. $\square$

<u>Next Time:</u> Curve Lifting Property.

<u>Remark:</u> Covering $\Leftrightarrow$ Surjective Local Homeomorphism + Curve Lifting Property

(so the curve lifting property can be tested on the cases we care about)