Lecture V: Curve Lifting Property, Covers, Degree of proper holomorphic maps

Last Time Topology interlude

$$\begin{array}{ccc} & & X \\ & & \downarrow p \\ Z & \xrightarrow{f} & Y \end{array}$$

X, Y Hausdorff
p local homeo
$z_0 \in Z$, $x_0 \in X$ with $p(x_0) = f(x_0)$

Q1: $\exists\ \hat{f}: Z \to X$ cont with $p \circ \hat{f} = f$ with $\hat{f}(z_0) = x_0$?
(lifting of f relative to p) ↳ "initial conditions".

Q2: Uniqueness?

THM 1: Local liftings exist and are unique as germs at z_0. ($\hat{f} = (p|_U)^{-1} \circ f$ on some $z_0 \in W$ open in Z, $x_0 \in U$ & $p|_U : U \xrightarrow{\sim} V$)

THM 2: If Z is connected, we have at most 1 global lift with prescribed initial conditions

THM 3: If $f = H : [0,1] \times [0,1] \to Y$ is a homotopy & each $u_s(t) = H(t,s)$ has a lift with the same initial condition $\hat{u}_s(0) = \hat{a}$ (with $p(\hat{a}) = a = H(0,s)$ $\forall s$)
then, $\exists\ \hat{H} : [0,1] \times [0,1] \to Y$ lift with $\hat{H}(0,s) = \hat{a}$ & $\hat{H}(1,s) = \hat{H}(1,0)$ $\forall s$
Moreover $\hat{H}$ is a homotopy between $\hat{u}_0$ & $\hat{u}_1$.

§5.1. Curve Lifting Property:

Fix X, Y top spaces

Def: A continuous map $p: X \to Y$ has the "curve lifting property" if the following condition holds: For every curve $u: [0,1] \to Y$ & every point $x_0 \in X$ with $p(x_0) = u(0)$ $\exists\ \hat{u} : [0,1] \to X$ lift of u relative to p with $\hat{u}(0) = x_0$.

Obs: $\hat{u}$ will be unique if p is a local homeomorphism by Theorem 2 §4.3

Q: Why this property?

A: It ensures liftings of ANY cont function $f: Z \to Y$ exist if Z is nice.

Theorem 1: Assume X, Y are Hausdorff Top spaces & $p: X \to Y$ is a local homeo with the "curve lifting property". Fix a Top space Z which is simply connected & locally path connected. Then, for every $f: Z \to Y$ cont & every $z_0 \in Z$ & $x_0 \in X$ with $f(z_0) = p(x_0)$ $\exists !$ lift $\hat{f}: Z \to X$ of f relative to p with $\hat{f}(z_0) = x_0$.

Proof Since Z is connected & locally path connected, then Z is pathwise connected by Lemma 2 §3.3. Pick $z \in Z$ & a curve $u: [0,1] \to Z$ on Z joining z_0 & z. Then, the

curve $v = f \circ u : [0,1] \to X$ joins $f(z_0) = y_0$ & $f(z) = y$.

By hypothesis, v can be lifted (uniquely) relative to p to a curve $\hat{v}: [0,1] \to X$ with $\hat{v}(0) = x_0$ (since $p(x_0) = v(0) = y_0$)

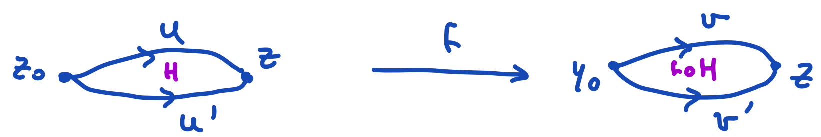
Define $\hat{f}: Z \longrightarrow X$ by $\hat{f}(z) = \hat{v}(1)$

Claim 1: The definition of $\hat{f}$ is independent of the choice of curve u. [Use: $\pi_1(Z) = \{0\}$ & p local homeo]

Pf/. Pick a different curve u' on Z joining z_0 to z. Then, $u \sim u'$ because $\pi_1(Z, z) = \{0\}$

- Fix $H: [0,1] \times [0,1] \longrightarrow Z$ homotopy between u & u'
- Pick $v' = f \circ u'$ & its unique lift to $\hat{v}': [0,1] \to X$ with $\hat{v}'(0) = x_0$.

Then $v \sim v'$ via the homotopy $A = f \circ H : [0,1] \times [0,1] \longrightarrow Z$



By the "curve lifting property", each curve $v_s(t) := A(s,t)$ on Y lifts relative to p to a (unique) curve $\hat{v}_s$ with $\hat{v}_s(0) = \hat{v}(0) = x_0$ <u>Note:</u> $\hat{v}_0 = \hat{v}$ & $\hat{v}_1 = \hat{v}'$

Theorem 1 §4.4 (p local homeo) ensures the whole homotopy lifts relative to p to $\hat{A}: [0,1] \times [0,1] \to X$ with $\hat{A}(t,s) = \hat{v}_s(t)$. In particular $\hat{A}(1,s) = \hat{A}(1,0) = \hat{v}_0(1) = \hat{v}(1)$ for all s so $\hat{v}'(1) = \hat{A}(1,1) = \hat{v}(1)$ <u>Conclusion:</u> $\hat{f}$ is independent of u.

Claim 2: $\hat{f}$ is continuous [Use: Z is locally pathwise connected & p local homeo]

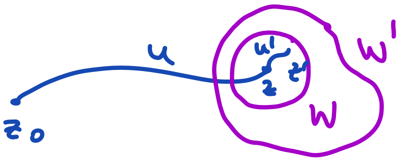
Pf/ Pick $z \in Z$ & $x = \hat{f}(z)$. Fix a neighborhood U_0 of x in X.

We want to find $W \subseteq Z$ open with $z \in W$ so that $\hat{f}(W) \subseteq U_0$. ($\Rightarrow \hat{f}^{-1}(U_0)$ would be open in Z)

Since p is a local homeo, we can find $U' \subseteq U_0$, $V' \subseteq Y$ with $x \in U'$ & $p(x) = f(z) \in V'$ with $p|_{U'} : U' \longrightarrow V'$ homeo.

Next, $W' := f^{-1}(V')$ open in Z & $z \in W'$. Since Z is locally pathwise connected $\exists\ \boxed{W} \subseteq W'$ open & pathwise connected with $z \in W$.

We need to show $\hat{f}(W) \subseteq U_0$, ie $f(z') \in U_0 \ \forall z' \in W$.



Pick a path u' in W joining z & z'.

Then $u * u'$ is a path in Z joining z_0 to z'. The curve $v' = f \circ u'$ lies entirely in V', with $v'(0) = f(z)$.

The curve lifting property & local homeo condition ensures v' lifts uniquely wrt p to a curve $\hat{v}'$ with $\hat{v}'(0) = \hat{f}(z)$ ($\hat{v}' = (p|_U)^{-1} \circ v'$)

Then $\hat{v}$ & $\hat{v}'$ can be concatenated to a curve $\widehat{v * v'} = \hat{v} * \hat{v}'$ lifting $v * v' = f \circ (u * u')$ relative to p with $\hat{v} * \hat{v}'(0) = \hat{v}(0) = x_0$.

Thus, $\hat{f}(z') = \widehat{v * v'}(1) = \hat{v}'(1) \in U' \subseteq U_0$ $\square$

Remark: Counterexample if Z is not locally path connected?

Q2: Which continuous functions have the "curve lifting property"?

A: Coverings!

Def: A cont map $p: X \longrightarrow Y$ is a covering if $\forall y \in Y$ $\exists$ $V \subseteq Y$ open with $y \in V$ & a collection $\{U_j\}_{j\in J}$ of pairwise disjoint opens of X with

(1) $p^{-1}(V) = \bigsqcup_{j \in J} U_j$ $\quad (J = J_{(y)})$

(2) $p|_{U_j} : U_j \longrightarrow V$ is a homeomorphism

⚠ coverings are surjective!

Examples (1) $p: \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$, $z \longmapsto z^N$ is a covering map $\forall N$

$p^{-1}(z^N) = \{z, \omega z, \ldots, \omega^{N-1} z\}$ where ω is an N^{th} primitive root of 1

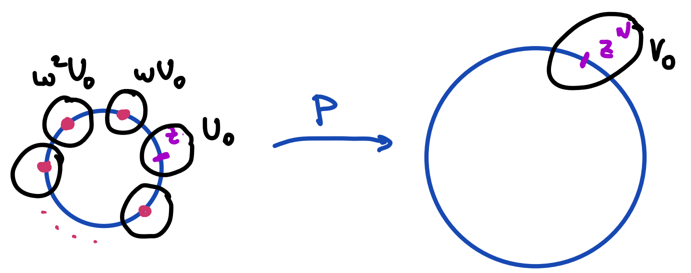
p is a local homeomorphism so $\exists$ U_0 open nbhd of z in $\mathbb{C}^\times$ with

$p|_{U_0}: U_0 \longrightarrow V_0$ local homeo.

$$(*) \qquad p^{-1}(V_0) = U_0 \cup \omega U_0 \cup \cdots \cup \omega^{N-1} U_0$$

Then; we can replace U_0 (& V_0 accordingly) by a disc $D_{(0,r)}$ with $r>0$ small enough

To ensure $\quad U_0 \cap \omega^j U_0 = \emptyset \quad \forall j$ in $\{1, \ldots, N-1\}$



Thus: (1) The collection $\{U_0, \omega U_0, \ldots, \omega^{N-1} U_0\}$ is pairwise disjoint and partitions $p^{-1}(V_0)$

(2) $p|_{\omega^j U_0} : \omega^j U_0 \xrightarrow{\sim} V_0$

(2) $\mathbb{D} \hookrightarrow \mathbb{C}$ is not a covering (it's not surjective!)

(3) $\exp: \mathbb{C} \longrightarrow \mathbb{C}^\times$ is a covering map. (use a local branch of log + fiber $\exp^{-1}(\exp z) = z + 2\pi i \mathbb{Z}$ $= z +$ rk-1 discrete lattice)

(4) $\mathbb{C} \longrightarrow \mathbb{C}/\Gamma$ is a covering map (same argument as (3))

Lemma 1: Covering maps are local homeomorphisms

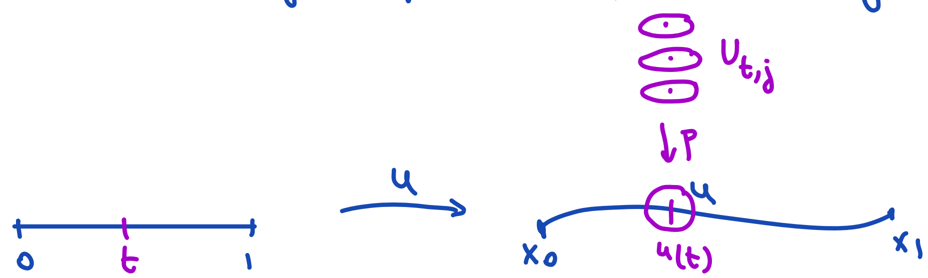
Pf/ Pick $x \in X$ & $p(x) = y$ Then $\exists\ y \in V \subseteq Y$ open & $\{U_j\}$ as in the definition. By (1), $\exists!\, j$ with $x \in U_j$. By (2), $p|_{U_j} : U_j \xrightarrow{\sim} V_j$ homeo.

Theorem 2: Every covering map $p: X \longrightarrow Y$ of topological spaces has the curve lifting property. Furthermore, liftings are unique when initial conditions are prescribed

Proof: Uniqueness follows by Lemma 1 §4.5 & Theorem 2 §4.3 (uniqueness of lifts relative to local homeomorphisms.

. **Existence:** Fix a curve $u: [0,1] \longrightarrow Y$ with $u(0) = y_0 \in Y$ & pick $x_0 \in X$ with $p(x_0) = y_0$

Glue local liftings of u on a convenient partition of $[0,1]$ with prescribed initial (concatenates) conditions (To ensure the concatenation can happen). The partition will be constructed using the fact that p is a covering map.



For each $t \in [0,1]$, we can find $V^{(t)} \subseteq Y$ open with $u(t) \in V^{(t)}$ & $\{U_j^{(t)}\}_{j \in J_t}$ opens in X with

(1) $p^{-1}(V_t) = \bigsqcup_{j \in J_t} U_j^{(t)}$ & (2) $p|_{U_j^{(t)}} : U_j^{(t)} \xrightarrow{\sim} V^{(t)}$ homeo $\forall j$

Since $u([0,1])$ is compact & $\{V_t\}_t$ covers $u_{[0,1]}$ $\exists\ t_1, t_2, \ldots, t_n$ with

$u([0,1]) = \bigcup_{i=1}^{n} V^{(t_i)}$ Relabel $V_i := V^{(t_i)}$

Next: Write $W_i := u^{-1}(V_i)$ for $i = 1, \ldots, n$

$\{W_i\}_{i=1}^{n}$ is a finite open cover of $[0,1]$, which is compact. This cover has a a Lebesgue number $\delta > 0$ ie whenever $t, s \in [0,1]$ satisfy $0 \le t - s \le \delta$, then $\exists\, i = 1, \dots, n$ with $[t, s] \subseteq W_i$ (see Proof of Prop 2 §3.3)

Conclusion: Fix $0 = s_0 < s_1 < s_2 < \dots < s_k = 1$ partition of $[0,1]$ of rat'l length $< \delta$

By allowing repetitions of V_i & relabeling if necessary, we have

(1) $u|_{[s_i, s_{i+1}]} \subseteq V_i$ for all $i = 0, \dots, k-1$

(2) $p^{-1}(V_i) = \bigsqcup_{j \in J_i} U_j^{(i)}$ $\quad (U_j^{(i)} = U_j^{(t_i)}$ & $J_i = J_{t_i})$

(3) $p|_{U_j^{(i)}} : U_j^{(i)} \xrightarrow{\sim} V_i$ local homeomorphism

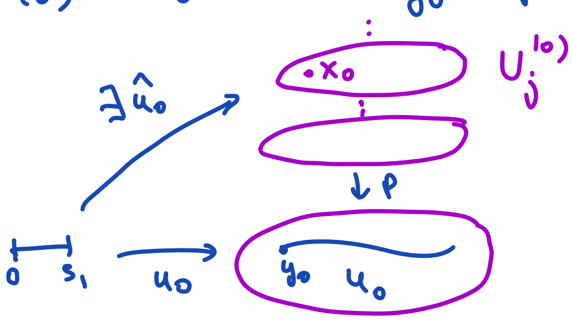
- Write $u_i = u|_{[s_i, s_{i+1}]}$ $\quad i = 0, \dots, k-1$ cont & $u = u_0 * \dots * u_{k-1}$

Claim: We can find ! lifts $\hat{u}_i$ of u_i with appropriate initial conditions with $\hat{u}_0(s_0) = x_0$, $\hat{u}_i(s_{i+1}) = \hat{u}_{i+1}(s_{i+1})$ $\forall\, i = 0, \dots, k-2$.

The concatenation $\hat{u} = \hat{u}_0 * \dots * \hat{u}_{k-1}$ will be the lift of u with prescribed initial value x_0.

Pf/ We proceed inductively.

- $i=0$ Since $u(0) \in V_0$ & $y_0 \in p^{-1}(V_0)$ we can find $j \in J_0$ with $y_0 \in U_j^{(0)}$.



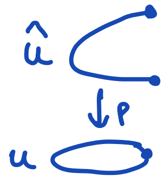
$\hat{u}_0 = (p|_{U_j^{(0)}})^{-1} \circ u_0$ is the desired lift.

- Assume we have constructed $\hat{u}_0, \dots, \hat{u}_i$ for $i \le k-2$ & set $y_i = u_i(s_{i+1}) \in V_i \cap V_{i+1}$

Since $\hat{u}_i(s_{i+1}) = x_i$ with $p(x_i) = y_i$, we can find $j \in J_{i+1}$ with $x_i \in U_j^{(i+1)}$. Then, $\hat{u}_{i+1} = (p|_{U_j^{(i+1)}})^{-1} \circ u_{i+1}$ lifts u_{i+1} with $\hat{u}_{i+1}(s_{i+1}) = x_i$ $\square$

Q: What happens to liftings of loops?

A: We need not get a loop!



Eg: $p : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$, $z \longmapsto z^2$, $x_0 = y_0 = 1$ & $u : [0,1] \to \mathbb{C}^\times$, $u(t) = e^{2\pi i t}$

Then $\hat{u} : [0,1] \longrightarrow \mathbb{C}$ is top-semicircle $\hat{u}(t) = e^{\pi i t}$ so $\hat{u}(1) = -1$.

However, this will <u>not</u> be the case if u is null-homotopic, precisely because homotopies ALWAYS lift relative to coverings (not just local homeos!) Furthermore, lifts will also be null-homotopic.

Proposition 1: Let X, Y be Hausdorff top. spaces & $p : X \to Y$ a local homeo with the curve lifting property (eg: a covering). Fix y_0 in Y & $x_0 \in X$ with $p(x_0) = y_0$ & let $u : [0,1] \longrightarrow Y$ be a loop based at y_0. If u is null-homotopic then any lift $\hat{u} : [0,1] \to Y$ of u relative to p with $\hat{u}(0) = x_0$ is also a loop. Furthermore, $\hat{u}$ is null-homotopic.

Pf/ Fix $H : [0,1] \times [0,1] \longrightarrow Y$ a homotopy between $\mathbb{1}_{y_0}$ & u.

Write $u_s = H(-, s)$. Note $u_s(0) = y_0$ $\forall s$.

By the curve lifting property, we have $\hat{u}_s : [0,1] \longrightarrow X$ lifting u_s relative to p with $\hat{u}_s(0) = x_0$. Furthermore $\hat{u}_0 = \hat{\mathbb{1}}_{y_0} = \mathbb{1}_{x_0}$. $\hat{u}_1 = \hat{u}$.

Thus, the hypothesis of Thm 1 §9.4 apply & $\hat{H}(t,s) = \hat{u}_s(t)$ is a homotopy $\hat{u}_0 = \mathbb{1}_{x_0}$ & $\hat{u}$. In particular, $\hat{u}(1) = \hat{u}_0(1) = x_0$. So $\hat{u}$ is a loop based at x_0 & it is null-homotopic in X. $\square$

This result has a direct consequence: fibers of coverings are in bijective with each other if Y is pathwise connected (ok for R.S.)

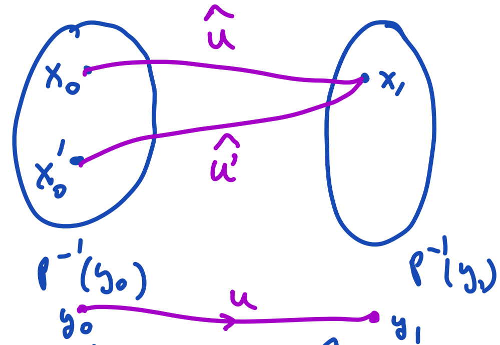
Corollary 1: Let X, Y be Hausdorff top. spaces & $p: X \to Y$ a local homeo with the curve lifting property (eg: a covering). If Y is pathwise connected, then $p^{-1}(y_0) \xrightarrow[\text{bij}]{\sim} p^{-1}(y_1)$ $\forall\, y_0, y_1 \in X$. The bijection is given by lifting a path from y_0 to y_1 with prescribed initial condition in $p^{-1}(y_0)$

Proof: Pick a curve $u: [0,1] \to Y$ joining y_0 & y_1 & fix $x_0 \in p^{-1}(y_0)$

Let $\hat{u}: [0,1] \to Y$ be the unique lifting of u relative to p with $\hat{u}(0) = x_0$

We define $\varphi: p^{-1}(y_0) \xrightarrow{\sim} p^{-1}(y_1)$

$$x_0 \longmapsto x_1 = \hat{u}(1)$$



- φ is well-defined (& depends on u)
- φ is injective: if $\hat{u}$ & $\hat{u}'$ are the liftings of u with $\hat{u}(0) = x_0$, $\hat{u}'(0) = x_0'$ for $x_0, x_0' \in p^{-1}(y_0)$ & $\hat{u}(1) = \hat{u}'(1)$, then $\hat{u}^-$ & $\hat{u}'^-$ are two distinct liftings of u^- with the same initial conditions. This cannot happen by the uniqueness property ($[0,1]$ is connected).
- Similar reasoning allows us to build $\psi: p^{-1}(y_1) \to p^{-1}(y_0)$ by working with u^- & confirming $\psi \circ \varphi = \varphi \circ \psi = id$, so φ is a bijection. $\square$

Corollary 2: If Y is pathwise connected & $p: X \to Y$ is a local homeo with the curve lifting property, then p is surjective.

Corollary 3: If $p: X \to Y$ is a covering & Y is pathwise connected, we conclude all fibers have the same size. We call it the <u>number of sheets</u> of the covering

§5.2. Coverings:

Q: How to check if $p: X \longrightarrow Y$ is a covering?

Necessary conditions: (1) surjection, (automatic from (2) & (3) if Y is connected)
(2) local homeomorphism
(3) curve lifting property

If we restrict our hypotheses on X & Y we get sufficient conditions:

Theorem 1: Assume X is Hausdorff, Y is a manifold & pick $p: X \longrightarrow Y$. cont & surj
TFAE: (1) p is a covering
(2) p is a local homeomorphism with the curve lifting property

Note: If Y is connected, surj of p follows by Corollary 2 §5.1 because Y will be path-conn.

Pf/ (1) $\Rightarrow$ (2) p is a local homeo by Lemma 1 §5.1. The path lifting property holds by Theorem 2 §5.1.

(2) $\Rightarrow$ (1) We know that p is surjective. Fix $y \in Y$ & label the fiber by J, ie $p^{-1}(y) = \{x_j : j \in J\}$. Since Y is a manifold, we have a chart (V_0, φ) around y with $\varphi: V_0 \xrightarrow{\sim} W_0 \subseteq \mathbb{R}^n$, W_0 open. Shrink W_0 to a ball around $\varphi(y)$ & consider the inclusion $W_0 \overset{f}{\hookrightarrow} Y$ & set $V = f(W_0)$. We think of W_0 as V & f as the inclusion.

Now, W_0 is locally path connected & simply connected so by Theorem 1 §5.1 f has a unique lift $\hat{f}_j$ w.r.t p with $\hat{f}_j(y) = x_j$

Set $\boxed{U_j = \hat{f}_j(V).}$ $\forall j \in J$.

Claim 1: U_j is open $\forall j \in J$ & $U_j \subseteq p^{-1}(V)$

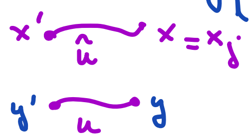
Pf/ Pick $x' \in U_j$ & $y' = p(x') \in V$, so $\hat{f}_j(y') = x'$. Pick open nbhds $U' \subseteq X$ & $V' \subseteq Y$ of x' & y' resp., with $p|_{U'}: U' \xrightarrow{\sim} V'$ homeo. Then, $\hat{f}_j$ agrees with the unique lift of f with initial value x' at y'! But locally, this map is given by $(p|_{U'})^{-1} \circ f$ which are both open, so $x' \in U' \subseteq U_j$. This confirms that $\hat{f}_j$ is open & $U_j \subseteq p^{-1}(V)$

Claim 2: $\{U_j\}_{j\in J}$ are pairwise disjoint.

Pf/ Pick $x' \in U_j \cap U_k$ for $j \neq k$. & set $y' = p(x')$. Then $\hat{f}_j$ & $\hat{f}_k$ agree (locally) with the lift of f with value x' at y'. Since V is connected, we conclude $\hat{f}_j = \hat{f}_k$ so $x_j = \hat{f}_j(y) = \hat{f}_k(y) = x_k$. Contradiction!

Claim 3 $p^{-1}(V) = \bigcup_{j\in J} U_j$.

Pf/ Inclusion ($\supseteq$) follows by Claim 1. Similarly, if $x' \in p^{-1}(V)$, set $y' = p(x')$. Since V is homeo to a ball, it is locally pathwise connected & simply connected. Then, we can pick a curve $u: [0,1] \to Y$ joining y' to y. We can lift this path to $\hat{u}: [0,1] \to X$ with $\hat{u}(0) = x'$. In particular $\hat{u}(1) = x_j$ for some j (since $\hat{u}(1) \in p^{-1}(y)$). Then, $\hat{u} = \hat{f}_j \circ u$, so $x' \in \hat{f}_j(u[0,1]) \subseteq U_j$.



Claim 4: $p|_{U_j} : U_j \xrightarrow{\sim} V$ is a homeo $\forall j$

Pf/ We know $p|_{U_j}$ is a local homeomorphism by construction combined with Claim 3.

- $p|_{U_j}$ is surjective: Use the same proof-technique of Claim 3. Indeed, pick $y \in V$ & a path $u: [0,1] \to Y$ joining y to y'. Pick the unique lift $\hat{u}_j$ of u relative to p with $\hat{u}_j(0) = x_j$. Then, $\hat{u}_j = \hat{f}_j \circ u$ by uniqueness & so $x' = \hat{u}_j(1)$ satisfies $p(x') = y'$.
- p is injective; say $x', x'' \in U_j$ satisfy $p(x') = p(x'')$ ($= y'$). Pick a path u joining y' to y. We lift it to 2 paths $\hat{u}'$ & $\hat{u}''$ with $\hat{u}'(0) = x'$ & $\hat{u}''(0) = x''$. But then $\hat{u}'(1), \hat{u}''(1) \in p^{-1}(y) \cap U_j = \{x_j\}$. So $\hat{u}'$ & $\hat{u}''$ agree with $\hat{f}_j \circ u^-$. We get $x' = \hat{f}_j(0) = x''$.
- Since $p|_{U_j}$ is bijective & a local homeomorphism, it is a homeomorphism.

Conclusion: By Claims 1 through 4, p is a covering. $\square$

§5.3 Proper Maps:

In the section we give a second sufficient condition for being a covering when X, Y are locally compact. Throughout, we assume X & Y are Hausdorff.

Def: Fix X a top space. We say X is <u>locally compact</u> (LC) if X is Hausdorff & every pt has an open neighborhood U with $\overline{U}$ compact (= "a relatively compact nbhd")

Equivalently, $\forall x \in X$ & U open nbhd of x $\exists V$ open nbhd of x with (1) $x \in V \subseteq U$ (2) $\overline{V} \subseteq U$ & $\overline{V}$ is compact

Obs: Riemann surfaces are LC (see Lemma 1 §5.4)

Def: Fix X & Y locally compact spaces & $p: X \longrightarrow Y$. We say p is <u>proper</u> if $p^{-1}(K)$ is compact in X for all $K \subseteq Y$ compact.

Remark: In Alg. Geom, $f: X \longrightarrow Y$ is proper if it's universally closed (1) & separated (2),

ie: (1)
$$\begin{tikzcd} X \times_Y Z \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\boxtimes"] & X \arrow[d] \\ Z \arrow[r, "f"] & Y \end{tikzcd}$$
p_1 is closed $\forall f: Z \to Y$ cont

(2) $\Delta \subseteq X \times_Y X$ is closed

If X, Y are locally compact, these conditions will be satisfied if $p^{-1}(K)$ is compact when $K \subseteq Y$ is. [See Stacks Project §5.17]

Lemma 1: If X is compact & Y is locally compact, then any $p: X \to Y$ cont is proper.

Pf/ Since Y is Hausdorff, any $K \subseteq Y$ compact is closed. So $p^{-1}(K) \subseteq X$ is closed in a compact space, hence it is compact.

Lemma 2: Proper maps are closed.

Pf/ Pick $Z \subset X$ closed. We want to show $p(Z) \subseteq Y$ is closed. By Lemma 3 (below), it's enough to show $p(Z) \cap K$ is compact for all $K \subset Y$ compact.

But $p(Z) \cap K = p(\underbrace{Z \cap \underbrace{p^{-1}(K)}_{\text{compact}}}_{\text{compact (closed in a compact set)}})$, so it is compact because p is cont.

$\square$

Lemma 3: Assume X is locally compact & let $V \subset X$. Then,

V is closed $\iff$ $\forall K \subseteq X$ compact $V \cap K$ is compact

Pf/ ($\Rightarrow$) V is closed & K is compact (so closed by Hausdorff condition). Then, $V \cap K$ is closed in K compact, hence compact.

($\Leftarrow$) Pick $x \in \overline{V}$ & fix W open nbhd of x in X st $\overline{W}$ is compact ($x \in X = \underset{\text{open}}{U}$ & X is L.C.). Then $\overline{W} \cap V$ is compact.

Since $x \in \overline{V}$ $\exists$ net $x_\alpha \xrightarrow[\alpha]{} x$ with $x_\alpha \in V$ $\forall \alpha$. But $x \in W$ open so $\exists$ subnet $x_{\alpha_i} \subseteq W$, which converges to x by the Hausdorff condition.

Now $x_{\alpha_i} \in W \cap V \subseteq \overline{W} \cap V$ compact so $x_{\alpha_i} \longrightarrow x \in \overline{W} \cap V$.

We conclude that $x \in V$, so V is closed. □

Theorem 1: If X, Y are L.C & $f: X \to Y$ is proper & discrete (ie fibers are discrete), then:

(1) $f^{-1}(y)$ is finite $\forall y \in Y$

(2) For each $y \in Y$ & $U \subseteq X$ open neihborhood of $f^{-1}(y)$, $\exists V \subseteq Y$ open nbhd of y with $f^{-1}(V) \subseteq U$.

Pf/ (1) is easy $f^{-1}(y)$ is compact & discrete so it must be finite.

(2) $f(X \setminus U)$ is closed by Lemma 2 §5.3. Since $y \notin f(X \setminus U)$ (because $U \supseteq f^{-1}(y)$), we can find $V \subseteq \underbrace{Y \setminus (f(X \setminus U))}_{\text{open}}$ open with $y \in V$. Then $V \cap f(X \setminus U) = \emptyset$, and so $f^{-1}(V) \subseteq U$. □

The next result gives another criteria for being a covering map if X, Y & p are special

Theorem 2: Fix $p: X \to Y$ a proper map with X, Y L.C. Assume p is surjective.

Then: p is a local homeomorphism $\iff$ p is a covering map.

Pf/ $(\Leftarrow)$ is Lemma 1 §5.1

$(\Rightarrow)$ Fix $y \in Y$. By Theorem 1 §5.3, $p^{-1}(y)$ is finite say $p^{-1}(y) = \{x_1, \ldots, x_n\} \neq \emptyset$

Furthermore since p is a local homeo $\exists\ x_j \in W_j$ open & $y \in V_j$ open with

$p|_{W_j}: W_j \xrightarrow{\sim} V_j$ homeo.

Since $p^{-1}(y)$ is discrete, we may assume $W_j \cap W_k = \emptyset$ if $j \neq k$

(Reason: We can shrink W_j so that $W_j \cap p^{-1}(y) = \{x_j\}$ (discrete fiber) & $x_k \notin \overline{W_j}\ \forall k \neq j$. (use X is LC)

Then, replace W_j by $\tilde{W}_j = W_j \setminus \bigcup_{k \neq j} \overline{W_k}$ open)

Next: $W = W_1 \cup \cdots \cup W_n$ is an open nbhd of $p^{-1}(y)$ so by Thm 1 (2) §5.3 $\exists\ V \subseteq Y$ open with $p^{-1}(V) \subseteq W$. By shrinking V further if necessary, we may assume (*) $\boxed{V \subseteq \bigcap_{j=1}^{n} p(W_j)}$ ($p(W_j) = V_j$ is open $\forall j$ & $y \in \bigcap_{j=1}^{n} p(W_j)$)

Set $U_j := W_j \cap p^{-1}(V)$ for $j = 1, \ldots, n$

Then (1) U_j are open & pairwise disjoint. , $p(U_j) = V$ by (*)

(2) $p^{-1}(V) = \bigcup_{j=1}^{n} U_j$

(3) $p|_{U_j}: U_j \xrightarrow{\sim} V$ homeo because $U_j \subseteq W_j$

Conclude: p is a covering. □

Remark 2: Surjectivity is necessary in Theorem 2 (hypothesis is missing in Theorem §4.22 of Forster)
One way to ensure it is to have Y connected, since then $p(X)$ will be closed (because of Lemma 2) & open by the local homeomorphism, hence $p(X) = P$.

§5.3 Degree of proper maps on R.S.

Our next goal is to show that non-constant holomorphic maps $f: X \to Y$ have a well-defined degree if f is proper.

Lemma 1: Riemann surfaces are locally compact

Pf/ Pick $x \in X$ RS & $x \in V$ open nbhd. Pick a coord. chart (U, φ) at x with $U \subseteq V$ & set $W = \varphi^{-1}(B(0, 1/2))$ if $\varphi : U \to \mathbb{D}$

Then $\overline{W} \simeq \overline{B(0, 1/2)}$ is compact & $x \in W \subseteq U$. X is Hausdorff by def of RS.

Setup: $f: X \to Y$ proper, non-constant holomorphic map ($\Rightarrow$ discrete)

Recall $A = \{x \mid f$ is not locally inj at $x\}$ = set of branch pts of f

$B := f(A)$ = set of critical values of f

Conclude:
- A is discrete & closed (p holomorphic)
(Consequence §4.1)
- B is closed & discrete because p is proper (use Theorem 1 (2) §5.2)

Set $X' := X \setminus f^{-1}(B) \subseteq X \setminus A$ & $f|_{X'} : X' \to Y'$

$Y' := Y \setminus B$

Lemma 2: (1) $f^{-1}(B)$ is closed and discrete

(2) X' & Y' are R.S.

(3) $f|_{X'}$ is a surjective, proper, unbranched holomorphic map
($\Rightarrow f|_{X'}$ is a covering map)

Pf/ (1) We know $f^{-1}(b)$ is discrete $\forall\, b \in B$.

Pick $c \in f^{-1}(B)$ & write $b = f(c)$. Pick $V \ni b$ open with $V \cap B = \{b\}$
(B is discrete)

- If $c \notin A$, then f is a local homeo near c, so we can shrink V to $V' \ni b$ & find $U' \ni c$ open with $f|_{U'} : U' \to V'$ homeo.

Now $f^{-1}(B) \cap U' = f^{-1}(b) \cap U' = \{c\}$.

$\hookrightarrow$ choice of $V' \subseteq V$

- If $c \in A$, $\exists$ U open with $c \in U$ & $U \cap A = \{c\}$ (A is discrete)

Write $W = f^{-1}(V) \cap U \Rightarrow f^{-1}(B) \cap W = f^{-1}(b) \cap W$

Since $f^{-1}(b)$ is finite (discrete & compact) & X is Hausdorff, we can shrink W to $W' \ni c$ so that $f^{-1}(b)\cap W' = \{c\}$. Conclude $f^{-1}(B)\cap W' = \{c\}$.

(2) It's enough to show that if Z is a R.S & $F\subseteq Z$ is discrete & closed, then $Z\setminus F$ is a R.S. It's open in a Hausdorff space, so $Z\setminus F$ is Hausdorff. Similarly, the restriction of the mxl atlas Σ to F is an atlas for $Z\setminus F$. To show that $Z\setminus F$ is connected, we prove it's pathwise connected.

Fix $z_0 \in Z\setminus F$ & let $z_1 \in Z\setminus F$. Since Z is pathwise connected (Lemma 2 §3.3), pick $u:[0,1] \longrightarrow Z$ curve joining z_0 & z_1.

Now $u([0,1])\cap F$ is discrete & closed in the compact set $u([0,1])$, so it's finite. Write it as $\{u(t_1)=y_1, \dots, u(t_s)=y_s\} \in Z$.

We can modify u by picking a coord chart $\begin{array}{c} U_i \xrightarrow{\sim} \mathbb{D} \\ y_i \longmapsto 0 \end{array}$ with $U_i\cap F = \{y_i\}$ & going around y_i instead of through y_i (U_i is null-homotopic)

The new curve u' lies in $Z\setminus F$, so $Z\setminus F$ is path connected.

(3) By construction $f|_{X'}$ is unbranched, holomorphic & non-constant. ($\overline{X\setminus f^{-1}(B)} = X$)

In particular $f|_{X'}$ is a local homeomorphism.

If we show $f|_{X'}$ is proper, surjectivity will follow from the connectedness of Y'! (see Remark 2 §5.2)

Pick $K\subseteq Y'$ compact. Then $K \subseteq Y$ is compact ($Y' \hookrightarrow Y$ is cont.).

But $f|_{X'}^{-1}(K) = \underbrace{f^{-1}(K)}_{\text{compact in } X} \cap \underbrace{X'}_{\text{open}} = f^{-1}(K)$ by construction, so it's compact.

Corollary 1: $\exists\, n\in \mathbb{Z}_{\geq 1}$ st $\forall\, x \in Y'$: $|f|_{X'}^{-1}(y)| = n$ (use Corollary 3 §5.1 + Lemma 2 §5.3)

Name: $n = \deg(f|_{X'})$ = "degree of f" ("generically, fibers of f have size n")

To finish, we'll show that $|f^{-1}(y)| = n$ for all $y \in Y$, if we count fibers over B appropriately. The branching numbers will do the trick.

- For $b \in B$, write $f^{-1}(b) = \{c_1, \dots, c_k\}$. We have a notion of local mult. (or branching near each c_i) Write $m_i = \nu(f, c_i)$ for $i = 1, \dots, k$.

(Locally, around c_i f behaves as $z \longmapsto z^{m_i}$.

Definition: $m = \nu(b) = \sum_{i=1}^{k} m_i$ degree at $b \in B$

($\nu(y) = 1$ if $y \in Y \setminus B$)

Theorem: If $f : X \longrightarrow Y$ is a non-constant proper holomorphic function and $\deg(f) = n$, then $\forall y \in Y$ $\nu(y) = n$ (" fibers have exactly n pts, counted with mult.")

Pf/ It's enough to prove this for $y = b \in B$. Pick a coord chart $(V, \varphi : V \xrightarrow{\simeq} \mathbb{D})$ for b with $V \cap B = \{b\}$. By Theorem 1(1) §5.2 $f^{-1}(b)$ is finite, say $f^{-1}(b) = \{c_1, \dots, c_k\}$

We know $f^{-1}(y)$ is discrete & compact, so finite $= \{c_1, \dots, c_k\}$

. After shrinking V if necessary, pick U_k coord chart for c_k in X so that

$$\begin{array}{ccc} f|_{U_j} : U_j & \longrightarrow & V \\ \rotatebox{90}{$\cong$} & & \rotatebox{90}{$\cong$} \\ \mathbb{D}' & \longrightarrow & \mathbb{D} \end{array}$$

behaves like $z \longmapsto z^{m_i}$ & $\boxed{U_k \cap f^{-1}(y) = m_i \ \forall y \in V \setminus \{b\}}$ $(*)$

By Theorem 1 (2) §5.2 $U_0 = \bigcup U_k$ is a nbhd of $f^{-1}(b)$ & we can find V' nbhd of b with $\boxed{f^{-1}(V') \subseteq U_0}$ $(**)$. Wlog, we assume $V' \subseteq V$.

Pick $y \in V' \setminus \{b\}$. Then $|f^{-1}(y)| = m_1 + \cdots + m_k$ by $(*)$ & $(**)$

But $y \notin B$ so $|f^{-1}(y)| = n = \deg(f)$. by Corollary 1 §5.3

Corollary 2: If X is a compact RS & $f : X \longrightarrow \mathbb{P}^1$ is non-constant holo, then $|f^{-1}(0)| = |f^{-1}(\infty)|$ (# zeroes of f = # poles of f, counted with mult.)