Lecture VI: Universal covers & Deck Transformations

Last Time: • Discussed "curve lifting property" & maps with this property (coverings!)

Why?

THM 1: If X, Y Hausdorff, $p: X \longrightarrow Y$ is a covering, & $f: Z \longrightarrow Y$ is cont with Z locally path connected & simply connected (eg: $Z = [0,1]$, $[0,1]\times[0,1]$), then f can be lifted rel. to p with any initial condition.

Corollary 1 • Homotopies lift relative to covers

• Null-homotopy curves in X will lift to null-homotopy curves in Y rel. to p.

THM 2: X, Y Hausdorff, Y path connected & $p: X \to Y$ covering then $p^{-1}(y_0) \xrightarrow{\text{bij}} p^{-1}(y_1)$ by lifting a path $\underset{y_0}{\bullet}\xrightarrow{u}\underset{y_1}{\bullet}$ with fixed $\hat{u}(0) = x_0 \in p^{-1}(y_0)$. $|p^{-1}(y_0)| = \#$ sheets of the covering.

Corollary: p is surjective

• Nice cases for testing coverings:

① **THM 3**: X Hausdorff, Y connected manifold & $p: X \to Y$. Then:

p covering $\Longleftrightarrow$ p is local homeo with curve lifting property

② **THM 4**: If X, Y are LC (so Hausdorff) & $p: X \longrightarrow Y$ is proper (eg: if X is compact) If p is surjective (eg if Y is connected), then p covering $\Longleftrightarrow$ p is local homeo

(Key: proper maps are closed)

THM 4: If X, Y are LC & $p: X \to Y$ is proper & discrete, then:

(1) $p^{-1}(y)$ is finite $\forall y \in Y$

(2) $\forall\ y \in Y$ & V open with $p^{-1}(y) \subseteq V$ $\exists$ U open in Y with $p^{-1}(U) \subseteq V$.

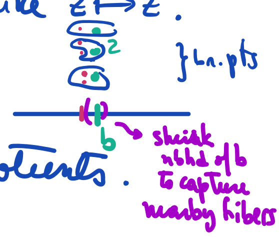
(Ex: p holo non-constant map on RS & X compact)

⇝ Used to check fibers of proper non-constant holo functions $f: X \to Y$ between R.S. are well-def

$$|f^{-1}(y)| = \sum_{\substack{x \in X \\ f(x)=y}} \nu(x,f)$$

where $\nu(x,f) = N$ if f around x behaves locally like $z \mapsto z^N$.

(Key: Branch pts, critical values & f^{-1}(crit values) are closed & discrete)



Next Goal: (1) Construct new RS from old ones via universal covers & quotients.

(2) Characterize these quotients via Deck transformations & Galois correspondence

§6.1 Universal Coverings:

Idea: Universal coverings are the largest covering maps to a fixed space Y. It will be unique up to unique iso. Its group of Deck Transformations will capture $\pi_1(Y)$. Universal coverings of R.S. will always exist.

Def: Fix X, Y connected top space & $p: X \to Y$ a covering.

We say p is the <u>universal covering of Y</u> if it satisfies the following universal property:

"For every $q: Z \to Y$ covering with Z connected & every $x_0 \in X$, $z_0 \in Z$ with $q(z_0) = p(x_0)$ $\exists$ exactly one continuous lift $\hat{p}: X \to Z$ relative to q with $\hat{p}(x_0) = z_0$"

$$\begin{tikzcd}
 & Z \ni z_0 \arrow[d, "q \text{ covering}"] \\
x_0 \in X \arrow[ur, dashed, "\exists! \hat{p}"] \arrow[r, "p"'] & Y
\end{tikzcd}$$

$q(z_0) = p(x_0)$

Lemma 1: Given Y, $\exists$ at most one univ covering up to unique isomorphism

Pf/ Say we have 2 univ covers $p: X \to Y$ & $p': X' \to Y$. Fix $y_0 \in Y$, $x_0 \in X$, $x_0' \in X'$ with $p(x_0) = y_0 = p'(x_0')$

By Univ Prop applied to p

$$\begin{tikzcd}
 & X' \arrow[d, "p'"] \arrow[dl, bend left, "\hat{p}'"] \\
X \arrow[ur, "\exists! \hat{p}"] \arrow[r, "p"'] & Y
\end{tikzcd}$$

with $\hat{p}(x_0) = \hat{p}'(x_0)$

Reversing the order, we find $\hat{p}'$ with $\hat{p}'(x_0') = x_0$.

Note: $\hat{p} \circ \hat{p}'$ works for

$$\begin{tikzcd}
 & X \arrow[d, "p' \circ \hat{p} = p \text{ covering}"] \\
X \arrow[ur, dashed] \arrow[r, "p"'] & Y
\end{tikzcd}$$

& $\hat{p} \circ \hat{p}'(x_0) = \hat{p}(x_0') = x_0$.

But id_X also works, so $\hat{p} \circ \hat{p}' = id_X$. Similar reasoning gives $\hat{p}' \circ \hat{p} = id_Y$. □

Q: How to know if a covering of a manifold Y by a manifold X is universal?

A: Check if the manifold X is simply connected!

Theorem 1: Assume X & Y are connected manifold, X is simply connected & $p: X \longrightarrow Y$ is a covering. Then p is the universal cover of Y.

Pf/ Since X is locally path connected & simply connected, then given any covering $q: Z \longrightarrow Y$ we can find a unique lift $\hat{p}: X \to Z$ rel to q given any initial conditions. This follows by Theorem 1 §5.1 & Theorem 2 §5.1 (coverings are local homeos with the curve lifting property). $\square$

Theorem 2: Suppose Y is Hausdorff, connected top space with a basis of path connected & simply connected opens (eg a connected manifold). Then $\exists\ \tilde{Y}$ Hausdorff, connected, simply connected top space & a covering map $p: \tilde{Y} \longrightarrow Y$.

Combining Thm 1 & 2 we get:

Corollary 1: Universal covers of connected manifolds exist!

Corollary 2: Universal covers of R.S exists & are RS & $p: \tilde{Y} \to Y$ is holomorphic for any R.S. Y.

Pf/ Use Corollary 1 & Theorem 2 §4.1.

Proof of Theorem 2: We build a Hausdorff top space $\tilde{Y}$ by constructing a basis & a natural projection map $p: \tilde{Y} \longrightarrow Y$. Then we show:
- p is a local homeo
- p has the curve lifting property
- $\tilde{Y}$ is simply connected.

Fix $y_0 \in Y$. Given $y \in Y$, write $\pi(y_0, y) = \{$ homotopy classes of $u: [0,1] \to Y$ with $u(0) = y_0$ & $u(1) = y \}$

$\pi(y_0, y) \neq \emptyset$ because Y is path connected.

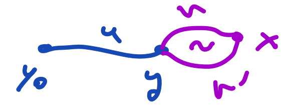
Define: $\boxed{\tilde{Y} = \{(y, \alpha) : y \in Y, \alpha \in \pi(y_0, y)\}}$ & $p = p_1 : \tilde{Y} \longrightarrow Y$, $(y, \alpha) \longmapsto y$

STEP 1: Build a basis for a topology on $\tilde{Y}$

Given $(y, \alpha) \in \tilde{Y}$ & $U \subseteq Y$ open nbhd of y which is path connected & simply connected, define a set $\mathcal{N}(U, y, \alpha) \subseteq \tilde{Y}$:

$\mathcal{N}(U, y, \alpha) = \{ (x, \beta) : x \in U \;\&\; \beta = [u * v]$ with $[u] = \alpha$ $u \in \Pi(y_0, y)$
$v : [0,1] \to U$ $v(0) = y$, $v(1) = x \}$

Note: β only depends on u because $\pi_1(U, y) = \{0\}$
[ ] : homotopy class.



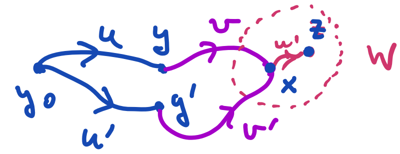
Set $\mathcal{B} = \{ \mathcal{N}(U, y, \alpha) \quad (y, \alpha) \in \tilde{Y} \;\&\; y \in U \subseteq Y$ open simply connected $\}$

Claim 1: $\mathcal{B}$ is a basis for a Topology on $\tilde{Y}$.

Pf/ (1) $\mathcal{B}$ covers $\tilde{Y}$ $\quad (y, \alpha) \in (U, y, \alpha)$.

(2) Finite intersection axiom: Fix $(x, \beta) \in \mathcal{N}(U, y, \alpha) \cap \mathcal{N}(U', y', \alpha')$.

so $\beta = [u * v]$ & $\beta = [u' * v']$
$[u] = \alpha$ $\quad [u'] = \alpha'$
$v \subseteq U$ $\quad v' \subseteq U'$



By the hypothesis on Y: $\exists$ $W \subseteq U \cap U'$ path conn. & simply connected open with $x \in W$

$\Rightarrow \mathcal{N}(W, x, \beta) \subseteq \mathcal{N}(U, y, \alpha) \cap \mathcal{N}(U', y', \alpha')$
$\quad \rotatebox{90}{$\in$}$
(z, γ) has $[\gamma] = [w \cdot w']$ $\quad [w] = \beta = [u \cdot v]$ $\quad v$ in U, w' in $W \subseteq U$
$\qquad \beta = [u' \cdot v']$ $\quad v'$ in U', w' in $W \subseteq U'$

$\Rightarrow w \sim \underbrace{u}_{\alpha} \cdot \underbrace{(v \cdot w')}_{y \to z \text{ in } U}$ $\quad w \sim \underbrace{u'}_{\alpha'} \cdot \underbrace{(v' \cdot w')}_{y \to z \text{ in } U'}$ $\quad$ so $\quad (z, \gamma) \subseteq [U, y, \alpha]$
$(z, \alpha) \subseteq [U', y', \alpha']$

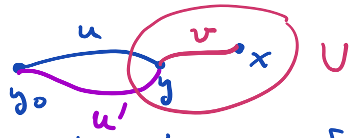
STEP 2: Check $\tilde{Y}$ is Hausdorff & path connected.

Claim 2: $\tilde{Y}$ is Hausdorff

Pf/ We analyze 2 cases:

- $(y, \alpha), (y', \alpha')$ $y \neq y'$ can be separated by $U \ni y$, $U' \ni y'$ disjoint path conn & simply connected in Y
  $\mathcal{N}(U, y, \alpha) \cap \mathcal{N}(U', y', \alpha') = \emptyset$
- $(y, \alpha), (y, \alpha')$ $\alpha \neq \alpha'$. Pick U simply connected open nbhd of y in Y.
  We show $\mathcal{N}(U, y, \alpha) \cap \mathcal{N}(U, y, \alpha') = \emptyset$, arguing by contradiction

Pick $(x,\beta)\in\mathcal{N}(U,y,\alpha)\cap\mathcal{N}(U,y,\alpha')$



so $\beta=[u\cdot v]=[u'\cdot v]$ for $[u]=\alpha$, $[u']=\alpha'$ & $v:[0,1]\to U$ joining y & x

$\Rightarrow \alpha=[u]=[u\cdot v\cdot v^-]=[u'\cdot v\cdot v^-]=[u']=\alpha'$ Contr!

Claim 3: $\tilde{Y}$ is path connected, hence connected.

Pick $(y_0,[1_{y_0}])$ as our base point in $\tilde{Y}$. Fix $(y,\alpha)\in\tilde{Y}$ with $[u]=\alpha$

$u:[0,1]\longrightarrow Y$ joining y_0 & y. We define a path in $\tilde{Y}$ joining $(y_0,[1_{y_0}])$ & (y,α):

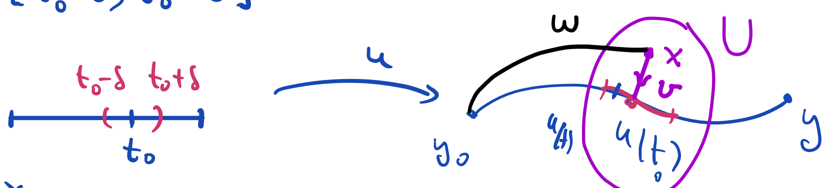
(*) $\boxed{\tilde{u}:[0,1]\longrightarrow\tilde{Y} \text{ with } \tilde{u}(t)=(u(t),[u_t]) \text{ where } u_t(s):=u(st)\ \forall s}$

Note u_t joins y_0 & $u(t)$

- Start & Endpoint of $\tilde{u}$: $\tilde{u}(0)=(y_0,[1_{y_0}])$, $\tilde{u}(1)=(u(1),[u])=(y,\alpha)$
- $\tilde{u}$ is continuous: given $\mathcal{N}(U,x,\gamma)$ nbhd of $\tilde{u}(t_0)$ we want to find $\delta>0$

st $\tilde{u}(t)\in\mathcal{N}(U,x,\gamma)\ \forall\ t\in[t_0-\delta,t_0+\delta]$

Pick δ st $u_{[t_0-\delta,t_0+\delta]}\subseteq U$



$\omega:[0,1]\longrightarrow Y$ $\omega(0)=y_0$, $\omega(1)=x$ $[\omega]=\gamma$

Since $\tilde{u}(t_0)\in\mathcal{N}(U,x,\gamma)$ $\exists\ v:[0,1]\longrightarrow U$ with $v(0)=x$, $v(1)=u(t_0)$

& $[u_{t_0}]=[\omega\cdot v]$

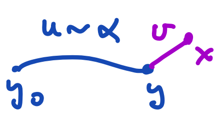
$\Rightarrow [u_t]=[u_{t_0}*u^-_{[t,t_0]}]=[\omega\cdot\overbrace{v\cdot u^-_{[t,t_0]}}^{\subseteq U}]\quad\forall\ t\in[t_0-\delta,t_0+\delta]$

so $\tilde{u}(t)\in\mathcal{N}(U,x,\gamma)\ \forall\ t\in[t_0-\delta,t_0+\delta]$.

Claim 4: $\varphi: p|_{\mathcal{N}(U,y,\alpha)}:\mathcal{N}(U,y,\alpha)\longrightarrow U$ is a homeomorphism ($\Rightarrow p$ is cont & a local homeomorphism)

Pf/ We build $\Psi: U\longrightarrow\mathcal{N}(U,y,\alpha)$

$x\longmapsto[x,[u*v]])$



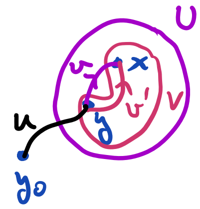
where v is a path from y to x.

The map is well-defined & $\Psi \circ \varphi = \varphi \circ \Psi = \mathrm{id}$.

• <u>φ is open</u> : any $\mathcal{N}(V, x, \beta) \subseteq \mathcal{N}(U, y, \alpha)$ maps to V open.

• <u>φ is cont</u> : given $W \subseteq U$ open, we can cover it with $(V_i)_{i\in I}$ open & simply connected, so it's enough to show $\varphi^{-1}(V_i)$ is open in $\mathcal{N}(U, y, \alpha)$. We have to analyze two cases:

(1) <u>If $y \in V_i$</u> then $\varphi^{-1}(V_i) = \mathcal{N}(V_i, y, \alpha) \subseteq \mathcal{N}(U, y, \alpha)$



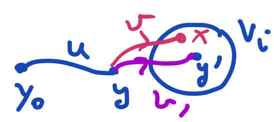
Why? $\varphi^{-1}(V_i) = \{(x, \beta)$ $x \in V_i \cap U = V_i$ & $\beta \in \pi_1(y_0, x)\}$
$\beta = [u * v]$ with $[u] = \alpha$
$v : y \longrightarrow x$ in U.

Now $V_i \subseteq U$ is path conn & simply connected, so we can find v' in $V_i \subseteq U$ joining y & x. Then $[v'] = [v]$ & $v'_{[0,1]} \subseteq V$. so (x, β) is in $\mathcal{N}(V_i, y, \alpha)$

Thus $\varphi^{-1}(V_i) = \mathcal{N}(V_i, y, \alpha)$ ($\supseteq$ is true by construction)

(2) <u>If $y \notin V_i$</u> then pick any $y' \in V_i$ & a path $v' : [0,1] \longrightarrow U$ $v'(0) = y$, $v'(1) = y'$.

Then $\varphi^{-1}(V_i) = \mathcal{N}(V_i, y', [u * v'])$



Why? $\varphi^{-1}(V_i) = \{(x, \beta)$: $x \in V_i \cap U = U_i$, $\beta \in \pi_1(y_0, x)$ with
$\beta = [u * v]$ with $[u] = \alpha$ & $v : y \longrightarrow x$ in $U\}$

$\beta = [u * v] = [(u * v') * w]$ for any path w in V_i joining y' to x

because U is simply connected. So $(x, \beta) \in \mathcal{N}(V_i, y', [u * v'])$

$\Rightarrow \varphi^{-1}(V_i) = \mathcal{N}(V_i, y', [u * v'])$ ($\supseteq$ is immediate)

<u>Claim 5</u>: $\tilde{Y}$ has the curve lifting property

Pf/ We break the argument into 3 cases:

① Lifting of a curve u with $u(0) = y_0$ with initial condition $\hat{u}(0) = (y_0, [1_{y_0}])$

This is the content of Claim 4 ($\hat{u} = \tilde{u}$)

② [This step is a special case of ③, so technically is not needed. However, I include it because it's enlightening]

Lifting of a curve u with $u(0)=y_0$ with $\hat{u}(0)=[y_0,\gamma]$ for $\gamma\in\pi_1(Y,y_0)$ (loop).

Say $\gamma=[v]$ for a loop v based at y_0

Take $\hat{u}_{(t)} = (u(t), [v * u_t])$

- <u>Initial condition:</u> $\hat{u}(0) = (u(0), [v * 1\!\!1_{y_0}]) = (y_0, [v]) = (y_0,\gamma)$
- <u>$\hat{u}$ is continuous:</u> Use the same argument as in Claim 3

Given $\hat{u}(t_0)\in \mathcal{N}(U,x,\beta)$, pick δ with $u(t_0-\delta, t_0+\delta)\subseteq U$.

Set $w:[0,1]\longrightarrow Y$ $w(0)=y_0$, $w(1)=x$ with $[w]=\beta$

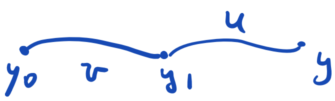
By def $[v*u_{t_0}] = [w * v']$ with $v':[0,1]\longrightarrow U$ $v'(0)=x$, $v'(1)=u(t_0)$

$\Rightarrow [v*u_t] = [v * u_{t_0} * \underbrace{u^-_{[t,t_0]}}_{\subseteq U}] = [w * \underbrace{v' * u^-_{[t,t_0]}}_{\subseteq U}]$ shows $\hat{u}_{(t_0-\delta, t_0+\delta)} \subseteq \mathcal{N}(U,x,\beta)$.

③ Lifting curves with arbitrary start and ending points.

- Pick $u:[0,1]\longrightarrow Y$ with arbitrary initial point $y_1 = w(0)$

Pick any $\beta\in\pi(y_0,y_1)$ & fix $v:[0,1]\to Y$ with $v(0)=y_0$, $v(1)=y_1$



<u>GOAL</u>: Lift u to $\hat{u}:[0,1]\longrightarrow \tilde{Y}$ with $\hat{w}(0) = (y_1,\beta)$

- We lift $u' = v * u$ using ① to a cont $\hat{u}':[0,1]\to\tilde{Y}$ with $\hat{u}'(0) = (y_0, [1\!\!1_{y_0}])$

$\Rightarrow u'(½) = y_1$ & $\hat{u}'(½) = (u'(½), [u'_{½}]) = (y_1, [v])$

Then $\hat{u}_{(t)} = \hat{u}'(\frac{t}{2}+\frac{1}{2})$ is cont & lifts u relative to p with the given initial conditions.

<u>Claim 6:</u> $\tilde{Y}$ is simply connected

Pf/ Pick a loop w on $\tilde{Y}$ based at $(y_0,[1\!\!1_{y_0}])$, then $u = p\circ w$ is a loop at $\tilde{Y}$ based at y_0.

Now take $(y_0, [1\!\!1_{y_0}])$ & using Claim 5, lift u relative to p to the

unique path $\hat{u}: [0,1] \longrightarrow \tilde{Y}$ with $\hat{u}(0) = (y_0, [\mathbb{1}_{y_0}])$. The uniqueness & (*) ensures that $w_{(t)} = \hat{u}_{(t)} = (u(t), [u_t])$. Thus $[\mathbb{1}_{y_0}] = [u_1] = [u]$ (take $t=1$).
So u is null-homotopic.

Since p is a covering & homotopies lift by Proposition 1 §5.1, we get $\hat{u} = w$ is null-homotopic. $\square$

§6.2 Deck Transformations

Fix X, Y Top spaces & $p: X \longrightarrow Y$ a covering map.

Definition: A Deck (or covering) transformation is a fiber-preserving homeomorphism $f: X \longrightarrow X$ (ie $p f(x) = p(x)$, equiv $f(p^{-1}(y)) = p^{-1}(y)$)

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & X \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

Remark 1: This set is a group under composition We denote it by $\mathrm{Deck}(X \xrightarrow{p} Y)$ or $\mathrm{Deck}(X|Y)$ if the choice of p is clear from context.

Our main case of interest: Y connected mfld (eg R.S) & $p: \tilde{Y} \longrightarrow Y$ univ cov.

Q1: $\mathrm{Deck}(\tilde{Y}|Y) = ?$

Q2: Pick any other covering $q: X \longrightarrow Y$ (X Hausdorff connected). Then

$$\begin{tikzcd} \tilde{y}_0 \in \tilde{Y} \arrow[rr, dashed, "\exists ! f"] \arrow[dr, "p"'] & & X \ni x_0 \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

$p(\tilde{y}_0) = q(x_0)$

Then • $f: \tilde{Y} \longrightarrow X$ is a covering

• $\mathrm{Deck}(\tilde{Y}|X) \underset{\text{subgroup}}{<} \mathrm{Deck}(\tilde{Y}|Y)$

Given $H < \mathrm{Deck}(\tilde{Y}|Y)$, can we find X Hausdorff & a covering $q: X \longrightarrow Y$ with $\mathrm{Deck}(\tilde{Y}|X) = H$?

This will lead to the notion of Galois coverings. (Next Time!)