Lecture VII: Deck Transformations & Galois correspondence

Last Time: We defined *universal coverings*: $p:\tilde{Y}\to Y$ covering s.t. $\forall q: Z\to Y$ covering & $\tilde{y}_0\in\tilde{Y}$ & $z_0\in Z$ with $p(\tilde{y}_0)=q(z_0)$ covering we have a $!\ f:\tilde{Y}\to Z$ lifting of p rel. to q with $f(\tilde{y}_0)=z_0$.

$$\begin{tikzcd} \tilde{y}_0 \in \tilde{Y} \arrow[r, dashed, "\exists! f"] \arrow[d, "p"'] & Z \ni z_0 \arrow[dl, "q \text{ covering}"] \\ Y & \end{tikzcd}$$

$f(\tilde{y}_0)=z_0$

Remark: UC are unique up to unique iso.

THM 1: If $p: X\to Y$ covering, X, Y connected manifolds & X simply connected $\Rightarrow p$ is the universal cover of Y.

THM 2: Universal coverings of connected manifolds exist & they are Hausdorff, connected & simply connected.

- Generalizations of THM 1 & 2: Can weaken the hypotheses on X, Y
  - THM 1: Enough to require
    - Y connected, Hausdorff top space
    - X simply connected & locally path connected
    - $p: X\to Y$ covering

(Same proof as Theorem 1 §6.1)

- THM 2: Enough to required: Y Hausdorff, connected with a basis of simply connected & path connected opens.

  Reason: The opens in the basis $\{\mathcal{N}(U,y,\alpha)\}_{U,y,\alpha}$ of $\tilde{Y}$ are path connected

  $(x,\beta)\in\mathcal{N}(U,y,\alpha)$ $\quad(\beta=[u*v]\quad v: y\to x \text{ in } U,\ [u]=\alpha)$ can be joined to (y,α)

  via $\hat{u}:[0,1]\longrightarrow\mathcal{N}(U,y,\alpha)\subseteq\tilde{Y}\qquad \hat{u}(t)=(v(t),[u*v_t])$

  $\hat{u}(0)=(v(0),[u*1_y])=(y,[u])=(y,\alpha)\quad;\quad\hat{u}(1)=(v(1),[u*v])=(x,\beta)$

  $\hat{u}=(p|_{\mathcal{N}(U,y,\alpha)})^{-1}\circ v$ so it's continuous.

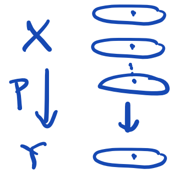
Corollary: Riemann surfaces admit universal coverings. Furthermore, $\tilde{Y}$ is a R.S & $p:\tilde{Y}\to Y$ is holomorphic.

Definition: A *Deck (or covering) transformation* of a covering $p: X\to Y$ is a fiber-preserving homeomorphism $f: X\to X$ (ie $pf(x)=p(x)$, equiv $f(p^{-1}(y))=p^{-1}(y)$)

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & X \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

We write this group (under comp) as $\mathrm{Deck}(X|Y)$ or $\mathrm{Deck}(X\xrightarrow{p}Y)$ if p is not clear from context

Remark: • f permutes "pancakes" over a connected open in Y



• Since X is connected, given $x_0, x_1 \in X$ with $p(x_0) = p(x_1)$
$\exists$ at most one $f \in \mathrm{Deck}(X|Y)$ with $f(x_0) = x_1$

Next goals: ① Compute $\mathrm{Deck}(\tilde{Y}|Y)$ $\rightsquigarrow$ Galois coverings

② Build quotients X of $\tilde{Y}$ & identify $\mathrm{Deck}(\tilde{Y}|X)$ with subgroups of $\mathrm{Deck}(\tilde{Y}|Y)$.
$\rightsquigarrow$ Galois correspondence

§7.1 Galois Coverings:

We are interested in special coverings, where points in any given fiber can be related via a Deck transf.

Definition: Assume X, Y are connected & Hausdorff. We say a covering $p: X \longrightarrow Y$ is Galois (alt. regular or normal) if given x_0, x_1 with $p(x_0) = p(x_1)$ $\exists$ $f \in \mathrm{Deck}(X|Y)$ with $f_{(x_0)} = x_1$

Remark 2: Since X is connected & f is a lift of p relative to p with fixed initial value f must be unique by Theorem 2 §4.3.

Examples: (1) $p: \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$ $p(z) = z^k$ is a covering map.

p is Galois: If $z^k = z'^k$, then $z = wz'$ for some $w \in \mathbb{C}^\times$ with $w^k = 1$
Then $f: \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$ $z \longmapsto wz$ is the corresponding element in $\mathrm{Deck}(\mathbb{C}^\times \xrightarrow{p} \mathbb{C}^\times)$
$\simeq \mathbb{Z}/k\mathbb{Z}$

(2) $p: \mathbb{H}^\ell \xrightarrow{\exp} \mathbb{D}^*$ where $\mathbb{H}^\ell = \{z : \mathrm{Re}(z) < 0\}$ is a Galois Covering
$\mathrm{Deck}(\mathbb{H}^\ell \longrightarrow \mathbb{H}^\ell) = \{\tau_{2\pi i n}(z) = z + 2\pi i n$ for $n \in \mathbb{Z}\} \simeq \mathbb{Z}$

The choice of terminology is not a coincidence:

Theorem 1: Assume Y is a connected manifold & let $p: \tilde{Y} \longrightarrow Y$ be its universal cover. ($\Rightarrow \tilde{Y}$ is a connected mfld). Then, p is Galois & $\mathrm{Deck}(\tilde{Y}|Y) \simeq \pi_1(Y)$.

Proof: We show both claims separated & build an explicit group isomorphism.

• **Claim 1:** p is Galois:

Pf/ Fix $x_0, x_1 \in \tilde{Y}$ with $p(x_0) = p(x_1)$. Fix $!f$ lifting of p relative to p with $f_{(x_0)} = x_1$

We need to show that f is a homeomorphism. We do this by explicitly building f^{-1} & showing it's continuous

$$\begin{tikzcd} & \tilde{Y} \ni x_1 \arrow[d, "p"] \arrow[dl, dashed, bend left, "\exists! g"] \\ x_0 \in \tilde{Y} \arrow[ur, dashed, "\exists! f"] \arrow[r, "p"'] & Y \end{tikzcd}$$

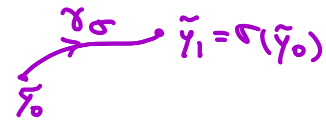
Similar reasoning builds! $g : \tilde{Y} \longrightarrow \tilde{Y}$ lifting of p relative to p with $g(x_1) = x_0$.

Thus $f \circ g,\ g \circ f : \tilde{Y} \longrightarrow \tilde{Y}$ lift p rel to p with $g \circ f(x_0) = x_0$ & $f \circ g(x_1) = x_1$.

We conclude: $f \circ g = g \circ f = id_{\tilde{Y}}$ by uniqueness of lifts ($\tilde{Y}$ is connected). $\bullet$

- Pick $\tilde{y}_0 = (y_0, \alpha) \in p^{-1}(y_0)$, eg $(y_0, [1\!\!1_{y_0}])$. We build an explicit map

$$\begin{aligned} \Phi : \mathrm{Deck}(\tilde{Y}\,|\,Y) &\longrightarrow \pi_1(Y, y_0) \\ \sigma &\longmapsto [p \circ \gamma_\sigma] \end{aligned}$$



where $\gamma_\sigma : [0,1] \longrightarrow \tilde{Y}$ is any curve with $\gamma_\sigma(0) = \tilde{y}_0$ & $\gamma_\sigma(1) = \sigma(\tilde{y}_0)$.

Φ is well-defined since $[\gamma_\sigma]$ is uniquely determined by the end points ($\tilde{Y}$ is simply connected) & $[p \circ v] = [p \circ v']$ if $[v] = [v']$ (if $v \sim v'$ via H, then $p \circ v \sim p \circ v'$ via $p \circ H$)

- <u>Claim 2</u>: Φ is a group homomorphism

Pf/ Let $\sigma_1, \sigma_2 \in \mathrm{Deck}(\tilde{Y}\,|\,Y)$ & pick curves γ_{σ_1} joining $\tilde{y}_0$ & $\sigma_1(\tilde{y}_0)$ in $\tilde{Y}$
γ_{σ_2} joining $\tilde{y}_0$ & $\sigma_2(\tilde{y}_0)$ in $\tilde{Y}$

Then $\sigma_1 \circ \gamma_{\sigma_2}$ joins $\sigma_1(\tilde{y}_0)$ & $\sigma_1 \circ \sigma_2(\tilde{y}_0)$ so $\gamma_{\sigma_1} * (\sigma_1 \circ \gamma_{\sigma_2})$ joins $\tilde{y}_0$ & $\sigma_1 \circ \sigma_2(\tilde{y}_0)$
$\Rightarrow = \gamma_{\sigma_1 \circ \sigma_2}$

Then $\Phi(\sigma_1 \circ \sigma_2) = [p \circ \gamma_{\sigma_1 \circ \sigma_2}] = [p \circ (\gamma_{\sigma_1} * \sigma_1 \circ \gamma_{\sigma_2})] = [(p \circ \gamma_{\sigma_1}) * (\underbrace{p \circ \sigma_1}_{=p} \circ \gamma_{\sigma_2})]$
$= [p \circ \gamma_{\sigma_1}] * [p \circ \gamma_{\sigma_2}] = \Phi(\sigma_1) * \Phi(\sigma_2)$ $\square$

- <u>Claim 3</u>: Φ is injective

Pf/ Fix σ with $\Phi(\sigma) = [1\!\!1_{y_0}]$. Then $[p \circ \gamma_\sigma] = [1\!\!1_{y_0}]$.

But $u = p \circ \gamma_\sigma$ is null-homotopic in Y & it has a unique lift to $\tilde{Y}$ with $\hat{u}(y_0) = (y_0, [1\!\!1_{y_0}]) = \tilde{y}_0$. By construction, $\hat{u}$ is a loop by Prop 1 §5.1 (it lifts a null-homotopic loop) Uniqueness of lifts gives $\hat{u} = \gamma_\sigma$ and $\tilde{y}_0 = \hat{u}(1) = \gamma_\sigma(1)$, so γ_σ is a loop around $\tilde{y}_0$. We get $\sigma(\tilde{y}_0) = \tilde{y}_0$. The uniqueness in Remark 2 gives $\sigma = id_{\tilde{Y}}$.

- <u>Claim 4</u>: Φ is surjective.

Pf/ Fix $\alpha \in \pi_1(Y, y_0)$ & v loop in Y based at y_0 with $[v] = \alpha$. Pick the unique lift $\hat{v}$ of v relative to p with $\hat{v}(0) = \tilde{y}_0$ & write $\tilde{y}_1 = \hat{v}(1)$, so $p(\tilde{y}_1) = y_0 = p(\tilde{y}_0)$,

Then, since p is Galois, we have $\sigma \in \mathrm{Deck}(\tilde{Y}|Y)$ with $\sigma(\tilde{y}_0) = \tilde{y}_1$.

We can take $\gamma_\sigma = \hat{v}$, so $\Phi(\sigma) = [p \circ \gamma_\sigma] = [p \circ \hat{v}] = [v] = \alpha$ $\square$

Examples: (1) $\exp : \mathbb{C} \longrightarrow \mathbb{C}^\times$ is the universal covering of $\mathbb{C}^\times$ because $\mathbb{C}$ is simply conn & exp is a covering (Theorem 1 §6.1)

- Given $n \in \mathbb{Z}$, write $\tau_n : \mathbb{C} \longrightarrow \mathbb{C}$ $\tau_n(z) = z + 2\pi i n$, homeo & $\exp(\tau_n(z)) = \exp(z)$, so $\tau_n \in \mathrm{Deck}(\mathbb{C}|\mathbb{C}^\times)$.
- For any $\sigma \in \mathrm{Deck}(\mathbb{C}|\mathbb{C}^\times)$ $\exp(\sigma(0)) = \exp(0) = 1$ so $\sigma(0) = 2\pi i n = \tau_n(0)$

$\Longrightarrow \sigma = \tau_n$ by uniqueness

Conclusion: $\pi_1(\mathbb{C}^\times) = \mathrm{Deck}(\mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times) = \{\tau_n : n \in \mathbb{Z}\} \simeq \mathbb{Z}$

(1') $\exp : \mathbb{H}^\ell \longrightarrow \mathbb{C}^\times$ is the universal covering of $\mathbb{C}^\times$ as well.

(2) $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ discrete rank-2 lattice in $\mathbb{C}$. Then, $\mathbb{C} \xrightarrow{\pi} \mathbb{C}/\Gamma$ is the universal covering of $\mathbb{C}/\Gamma$.

- Given $\gamma \in \Gamma$, pick $\tau_\gamma : \mathbb{C} \longrightarrow \mathbb{C}$ $\tau_\gamma(z) = z + \gamma$ homeo, $\pi(\tau_\gamma(z)) = \pi(z)$ $\forall \gamma$

so $\tau_\gamma \in \mathrm{Deck}(\mathbb{C}|\mathbb{C}/\Gamma)$.

- Given $\sigma \in \mathrm{Deck}(\mathbb{C}|\mathbb{C}/\Gamma)$, we get $\pi(\sigma(0)) = \pi(0) = 0 \in \mathbb{C}/\Gamma$

$\sigma(0) = \gamma = \tau_{\gamma}(0) \in \Gamma$ so $\sigma = \tau_\gamma$.

Conclusion $\pi_1(\mathbb{C}/\Gamma) = \mathrm{Deck}(\mathbb{C} | \mathbb{C}/\Gamma) = \{\tau_\gamma : \gamma \in \Gamma\} \simeq \Gamma \simeq \mathbb{Z} \times \mathbb{Z}$

[Consistent with $\mathbb{C}/\Gamma \underset{\text{homeo}}{\simeq} S^1 \times S^1$.]

§7.2 Galois correspondence:

Our next goal is to build covering maps from subgroup of $\mathrm{Deck}(\tilde{Y}|Y)$. This is the content of the "Galois correspondence". In order to do this, we'll need the notion of a proper discontinuous action on locally compact spaces (eg manifolds)

Definition: Fix X, Y top spaces, $p : X \to Y$ a covering & $H < \mathrm{Deck}(X|Y)$ (subgroup)

We say x, x' are equivalent modulo H if $\exists\, \sigma \in H$ with $\sigma(x) = x'$. (This defines an equivalence relation!) We write $x \sim_H x'$. ($\rightsquigarrow X/H := X/_{\sim_H}$ is a top space with quot top)

Theorem 1: X, Y connected manifolds,

$$\begin{tikzcd} \tilde{Y} \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[dl, "q"] \\ Y & \end{tikzcd}$$

univ covering p, q covering. Then:

(1) f is a covering map

(2) $H = \mathrm{Deck}(\tilde{Y} \xrightarrow{f} X) < \mathrm{Deck}(\tilde{Y}|Y)$ & $H \simeq \pi_1(X)$

(3) given a, a' in $\tilde{Y}$ we have: $f(a) = f(a') \iff a \sim_H a'$

Proof: We prove (1) by showing f is a local homeo with the curve lifting property (Using Theorem 1 §5.2). f is surjective because X is connected

Claim 1: f is a local homeomorphism.

$$\begin{tikzcd} a \in \tilde{Y} \arrow[r, "f"] \arrow[dr, "p"'] & X \arrow[d, "q"] \\ & Y \end{tikzcd}$$

$x = f(a)$

$b = f(a) = q(x)$

Pf/ We do so by f is locally a comp of homeomorphisms.

Pick $a \in \tilde{Y}$, $x = f(a) \in X$ & $y = p(a) = q(x)$. Since both p & q are local homeomorphism, we can find

$U \subseteq \tilde{Y}$ nbhd of a

$V, V' \subseteq Y$ nbhd of y

$W \subseteq X$ nbhd of x

so that $p|_U : U \xrightarrow{\sim} V$, $q|_{V'} : W \xrightarrow{\sim} V'$ homeos

Then $W' = q^{-1}(V \cap V') \subseteq X$ is open, $x \in W'$ & $q|_{W'} : W' \xrightarrow{\sim} V \cap V'$ homeo

$U' = p^{-1}(V \cap V') \subseteq Y$ is open, $a \in U'$ & $p|_{U'} : U' \xrightarrow{\sim} V \cap V'$ is homeo

Conclude: $f|_{U'} = q|_{W'}^{-1} \circ p|_{U'} : U' \xrightarrow{\sim} W'$ is homeo. ($=$ holds by uniqueness of lifts given $f(a) = x$)

Claim 2: f has the curve lifting property

Pf/ given a path u in Y & a point $a \in f^{-1}(u(0))$ we want to lift u to a curve $\hat{u}$ rel to f with $\hat{u}(0) = a$.

$$\begin{tikzcd} a \in \tilde{Y} \arrow[r, "f"] \arrow[dr, "p"'] & X \arrow[d, "q"] & {[0,1]} \arrow[l, "u"'] \arrow[dl, "v = q \circ u"] \\ & Y & \end{tikzcd}$$

We use $v = q \circ u : [0,1] \to Y$ path & $y = q(u(0))$. & $a \in p^{-1}(y)$.

Since p is a covering, we can lift v to $\hat{u}$ (uniquely) relative to p with $\hat{u}(0)=a$.

We need to check that $f\circ\hat{u}=u$.

Now, $f\circ\hat{u}$ & u both lift v relative to q. ($q\circ f\circ\hat{u} = p\circ\hat{u} = v = q\circ u$) & satisfy $f\circ\hat{u}(0) = f(a) = u(0)$. By uniqueness of lifts, we get $f\circ\hat{u}=u$ $\square$

- For (2): By construction any $h:\tilde{Y}\to\tilde{Y}$ homeo with $f\circ h=f$ satisfies $q\circ f\circ h = q\circ f$, so $p\circ h = p$ ie $h\in \mathrm{Deck}(\tilde{Y}\,|\,Y)$

Claim 3: $f:\tilde{Y}\longrightarrow X$ is the universal covering of X. ($\Rightarrow \pi_1(X)=\mathrm{Deck}(\tilde{Y}|X)$) because X is connected mfld

Pf 1/ By (1) $f:\tilde{Y}\longrightarrow X$ is a covering. In addition, $\tilde{Y}$ is simply connected because $p:\tilde{Y}\longrightarrow Y$ is the universal cover & Y is a connected manifold (use Theorem 2 §6.1)

Since X is a connected manifold, $f:\tilde{Y}\longrightarrow X$ is the universal covering of X (use Thm 2 §6.1)

Pf 2/ Use univ. prop for p to prove universal prop for f

$$\begin{tikzcd} a\in\tilde{Y} \arrow[r, dashed, "\exists! g"] \arrow[dr, "f"'] \arrow[ddr, bend right, "p"'] & Z\ni z \arrow[d, "q' \text{ covering}"] \\ & X \arrow[d, "q"] \\ & Y \end{tikzcd}$$

$f(a)=q'(z)$

$p(a)=(q\circ q')(z)$

Since $q\circ q'$ is a covering (compositions of coverings)

The univ prop of p gives us a unique $g:\tilde{Y}\to Z$ with $g(a)=z$ & $(q\circ q')\circ g = p$.

But $q'\circ g(a) = q'(z) = f(a)$ & both $q'\circ g$ & f satisfy the univ prop of p applied to the covering q. Uniqueness gives $f = q'\circ g$, as we wanted. $\square$

- For (3): ($\Leftarrow$) If $\sigma(a)=a'$ for some $\sigma\in H$, then $f(a) = f\circ\sigma(a) = f(a')$ $\quad H=\mathrm{Deck}(\tilde{Y}\xrightarrow{f}X)$

($\Rightarrow$) By Claim 3 & Theorem 1 §7.1 we have that $f:\tilde{Y}\longrightarrow X$ is Galois. Thus, if $a,a'\in\tilde{Y}$ with $f(a)=f(a')$ $\exists$ $\sigma\in\mathrm{Deck}(\tilde{Y}|X)=H$ with $\sigma(a)=a'$, ie $a\sim_H a'$. $\square$

This correspondence extends further:

Theorem 2: Given $H<\mathrm{Deck}(\tilde{Y}|Y)$, there exists a connected Hausdorff manifold X & covers $f:\tilde{Y}\longrightarrow X$, $q:X\longrightarrow Y$ with $p=q\circ f$ & $H=\mathrm{Deck}(\tilde{Y}\xrightarrow{f}X)$

To prove this, we need an interlude to fixed-pt free proper discontinuous actions.

Recall: G group & X a Top space. A group action $G \circlearrowright X$ is an action of G on the set $|X|$ st $g\cdot\_ : X \longrightarrow X$ is a continuous map for each $g\in G$.

Def: We say $G \circlearrowright X$ is properly discontinuous if for each $K\subset X$ compact the set $\{g\in G : g(K)\cap K\neq\emptyset\}$ is finite.

Equivalently, $\forall K_1, K_2$ compact: $\{g\in G : g(K_1)\cap K_2\neq\emptyset\}$ is finite (Pick $K=K_1\cup K_2$)

Examples: (1) $\mathbb{Z} \circlearrowright \mathbb{R}$ by translation.

(2) $SL_2(\mathbb{Z}) \circlearrowright \mathbb{H}=\{\operatorname{Im}(z)>0\}$ by linear fractional transf

• Another important example is the following:

Lemma 1: $\operatorname{Deck}(\tilde{Y}|Y) \circlearrowright \tilde{Y}$ is properly discontinuous. (Same is true for any subgroup)

Lemma 2: If X is Hausdorff, locally compact & $G \circlearrowright X$ is properly discontinuous, then $X/G = X/\sim_G$ (with the quotient top) is Hausdorff.

We discuss their proofs in §7.9 (below). Using these two results, we get Thm 2

Proof of Theorem 2: $H \circlearrowright \tilde{Y}$ is properly discontinuous by Lemma 1.

So we can construct $X=\tilde{Y}/H$ & $f=\pi_H:\tilde{Y} \longrightarrow\!\!\!\!\rightarrow X$ the quotient map

• Since f is continuous, and $\tilde{Y}$ is connected then so is X.

• By Lemma 2, X is Hausdorff.

• Next, we define $q: X\longrightarrow Y$ via $q(x)=p(a)$ if $f(a)=x$.

Claim 1: q is well-def

Pf/ Pick $x\in\tilde{Y}/H$ & $a, a'\in\tilde{Y}$ with $f(a)=x=f(a')$. This means $\exists\ \sigma\in H\subseteq \operatorname{Deck}(\tilde{Y}|Y)$ with $\sigma(a)=a'$. Then $p(a')=p\circ\sigma(a)=p(a)$, so q is well-defined.

Claim 2: q is continuous.

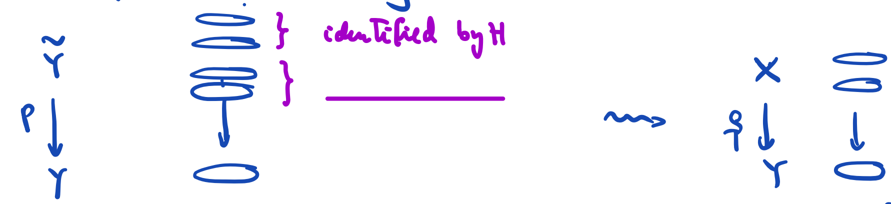
Pf/ Pick $U \subseteq Y$ open. Then: $g^{-1}(U) \subseteq X$ is open $\iff \pi^{-1}(g^{-1}(U)) \subseteq \tilde{Y}$ is open

$g^{-1}(U) = \{ x \in X$ with $g(x) \in U \} = \{ \pi(a) : a \in \tilde{Y}$ with $p(a) \subseteq U \}$

$\pi^{-1}(g^{-1}(U)) = \{ a \in \tilde{Y}$ with $p(a) \subseteq U \} = p^{-1}(U)$ & this set is open in $\tilde{Y}$.

<u>Claim 3</u>: g is a covering

Pf/



Given $y \in Y$, pick $y \in V \subseteq Y$ open & $\{U_j\}_{j \in J}$ opens in $\tilde{Y}$ with

(1) $p^{-1}(V) = \bigsqcup_{j \in J} U_j$ & (2) $p|_{U_j} : U_j \xrightarrow{\sim} V$ homeo $\forall j$

Take $W_j = f(U_j)$ $\forall j$ & remove repetitions to get a collection $\{W_j\}_{j \in J'}$ for some $J' \subseteq J$. By construction W_j is open since $\pi^{-1}(W_j) = \bigsqcup_{s \in S_j} U_s$, where $S_j := \{ k \in J \mid \exists\, \sigma \in H$ with $\sigma(U_k) = U_j \}$.

In addition, $g|_{W_j} : W_j \longrightarrow V$ is homeo $\forall j$ since $g|_{W_j}^{-1} = \pi \circ (p|_{U_j})^{-1}$ is continuous. □

<u>Claim 4</u>: X is a manifold & f is a covering

Pf/ Pull back the manifold structure of Y via the local homeomorphism g to turn X into a manifold. Then, we can use Theorem 1 (1) §7.2 & Claim 3 to conclude that f is a covering.

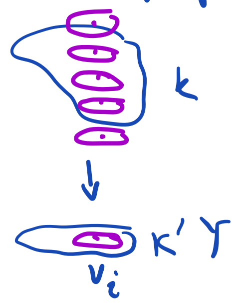
• Using Thm 1 (2), (3) §7.2 & Claims 1 through 4 we conclude that $H = \mathrm{Deck}(\tilde{Y} | X)$. □

§7.3. Proof of Lemmas 1 & 2 §7.2

<u>Lemma 1</u>: $\mathrm{Deck}(\tilde{Y}|Y) \circlearrowright \tilde{Y}$ is properly discontinuous.

<u>Proof</u>: We know $G = \mathrm{Deck}(\tilde{Y}|Y) \circlearrowright \tilde{Y}$ by definition. To check it's properly disc., we want to show that $|\{ g \in G : gK \cap K \neq \emptyset \}| < \infty$ for each $K \subseteq \tilde{Y}$ compact.

Take $K' = p(K) \subset Y$ compact. For each $y \in K'$ pick $V_y \subseteq Y$ open & $\{U_j^{(y)}\}_{j \in J_y}$ from the definition of covering. Since K' is compact, we can pick a finite set $\{V_1, \dots, V_n\}$ covering K' & the corresp collections of opens in $\tilde{Y}$ $\{U_j^{(i)}\}_{j\in J_i}$. These cover K, so we can restrict to a finite number of opens of $\tilde{Y}$ covering K.



The only $g\in G$ with $gK\cap K\neq\emptyset$ are the ones stabilizing one of the subcollections $\bigcup_{k=1}^{m_i} U_{j_k}^{(i)}$ for a fixed index $i=1,\dots,n$.

By the univ property of $\tilde{Y}\longrightarrow Y$, each pair $U_{j_k}^{(i)}, U_{j_s}^{(i)}$ $k\neq s$ is permuted by a single $g\in G$, so the stabiliser of each $\bigcup_{k=1}^{m_i} U_{j_k}^{(i)}$ is finite. This implies that $G\circlearrowright\tilde{Y}$ is properly discontinuous. $\square$

Lemma 2: If X is locally compact, Hausdorff & $G\circlearrowright X$ is properly discontinuous, then X/G (with the quotient top) is Hausdorff.

Pf/ We show the statement by proving any $x\neq x'$ in X can be separated by two G-invariant open nbhds $U\ni x$ & $U'\ni x'$ (G-invariant means $U=\pi^{-1}(\pi(U))$)

- Claim 1: We can reduce to the case when U is open & U' is G-invariant.

Pf/ Since $U\cap U'=\emptyset$, given any $g\in G$ we have $\emptyset=g\cdot\emptyset=gU\cap gU'=gU\cap U'$.

We then get $W=G\cdot U$ is open, G-invariant, $x\in W$ & $W\cap U'=\emptyset$. $\square$

- We start by picking U & V opens separating x & x' by the Hausdorff condition on X.

Using the local compactness we find opens U_1, V_1 with $\overline{U_1}$ & $\overline{V_1}$ compact & $x\in U_1\subseteq\overline{U_1}\subseteq U$
$x'\in V_1\subseteq\overline{V_1}\subseteq V$

- Claim 2: We have $U_1\cap gV_1=\emptyset$ for all but finitely many $g\in V$.

Pf/ Since $K_1=\overline{U_1}$ & $K_2=\overline{V_1}$ are compact & $G\circlearrowright X$ is properly disc, we have finitely many elements $g_1,\dots,g_s\in G$ with $K_1\cap g_iK_2=\emptyset$. But this implies

$U_1\cap gV_1=\emptyset \quad \forall g\notin\{g_1,\dots,g_s\}$. $\square$

By Claim 2, we can further shrink U_1 to ensure the remaining finitely many intersections are empty. Indeed, since $g\overline{V_1}$ is compact & X is Hausdorff, then $g\overline{V_1}$ is closed. Since $x \notin g\overline{V_1}$ we can shrink U_1 to get $U_1 \cap g\overline{V_1} = \emptyset$ for each $g \in \{g_1, \dots, g_s\}$.

This gives U_1 & $U' = \bigcup_{g \in V} g V_1$ as the neighborhoods of x & x' required in Claim 1.