# Lecture VIII: (Pre) Sheaves & Topology on set of stalks

So far, we've seen one of the constructions of R.S. from old ones (use universal covers & subgroups of $\mathrm{Deck}(\tilde{Y}|Y)$ for $Y$ = R.S.) This was construction 4 on Lecture 1

Next 2 constructions involve sheaves on $Y$ (holomorphic fncs & 1-forms) & analytic continuation of holomorphic germs along paths. The former is the topic of today's lecture.

**Recall**: Last time we discussed Deck Transformations & Galois correspondence.

**Theorem 1**: Assume $Y$ is a connected manifold & let $p\colon \tilde{Y} \longrightarrow Y$ be its universal cover. Then, $p$ is Galois & $\mathrm{Deck}(\tilde{Y}|Y) \simeq \pi_1(Y)$.

↳ $\forall\, a, a' \in \tilde{Y}$ with $p(a) = p(a')$ : $\exists\, \sigma \in \mathrm{Deck}(\tilde{Y}|Y)$ with $\sigma(a) = a'$

**Theorem 2**: Fix $X, Y$ connected manifolds, with

$$\begin{tikzcd} \tilde{Y} \arrow[r, "f"] \arrow[d, "p"', "\text{univ covering}"] & X \arrow[dl, "q \text{ covering}"] \\ Y & \end{tikzcd}$$

. Then:

(1) $f$ is a covering map

(2) $H = \mathrm{Deck}(\tilde{Y} \xrightarrow{f} X) < \mathrm{Deck}(\tilde{Y}|Y)$ & $H \simeq \pi_1(X)$

(3) given $a, a'$ in $\tilde{Y}$ we have: $f(a) = f(a') \iff a \sim_H a'$ ($\exists\, \sigma \in H$ with $\sigma(a) = a'$)

**Theorem 3**: Given $H < \mathrm{Deck}(\tilde{Y}|Y)$, there exists a connected Hausdorff manifold $X$ & covers $f\colon \tilde{Y} \longrightarrow X$, $q\colon X \longrightarrow Y$ with $p = q \circ f$ & $H = \mathrm{Deck}(\tilde{Y} \xrightarrow{f} X)$

Idea: $X = \tilde{Y}/H$, $f = \pi$ & $q(x) = p(a)$ if $\pi(a) = x$.

**Exercise**: The covering $q\colon X \longrightarrow Y$ is Galois $\iff G = \mathrm{Deck}(\tilde{Y}|X) \trianglelefteq \mathrm{Deck}(\tilde{Y}|Y)$.
If so, $\mathrm{Deck}(X \xrightarrow{q} Y) \simeq \mathrm{Deck}(\tilde{Y}/Y)/G$.

## §8.1 Application: Coverings of $\mathbb{D}^*$

Next, we classify coverings of $\mathbb{D}^* = \mathbb{D} \smallsetminus \{0\} = \{z \in \mathbb{C} : 0 < |z| < 1\}$. These coverings will be unbranched holomorphic maps by Corollary 2 §6.1. Since $\mathbb{D}^*$ is pathwise connected, all fibers of $f\colon X \longrightarrow \mathbb{D}^*$ unbranched covering will have the same size (by Corollary 3 §5.1) The classification depends on the number of sheets of the covering.

**Theorem 1**: Suppose $f : X \longrightarrow \mathbb{D}^*$ is an unbranched holomorphic covering. Then, $f$ is either isomorphic to the covering given by exp or by the $n^{th}$ power.

More precisely:

(1) If $f$ has infinitely many sheets, then $\exists\, \varphi : X \longrightarrow \mathbb{H}^\ell = \{z : \mathrm{Re}(z) < 0\}$ biholomorphic st

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & \mathbb{H}^\ell \arrow[dl, "\exp"] \\ & \mathbb{D}^* & \end{tikzcd}$$

(2) If $f$ has $N$ sheets, then $\exists\, \varphi : X \longrightarrow \mathbb{D}^*$ biholomorphic st

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & \mathbb{D}^* \arrow[dl, "p_N"] \\ & \mathbb{D}^* & \end{tikzcd}$$

where $p_N(z) = z^N$

**Proof**: We use the universal covering $\mathbb{H}^\ell \xrightarrow{\exp} \mathbb{D}^*$. Since $f$ is a covering we have

$$\begin{tikzcd} \mathbb{H}^\ell \arrow[rr, "\psi"] \arrow[dr, "\exp"'] & \circlearrowleft & X \arrow[dl, "f"] \\ & \mathbb{D}^* & \end{tikzcd}$$

$\psi$ is a holomorphic map by construction.

$\mathrm{Deck}(\mathbb{H}^\ell \mid \mathbb{D}^*) \simeq \mathbb{Z}$ by Example §7.1.

Now: by Theorem 1 §7.2, $\psi$ is a covering, $G = \mathrm{Deck}(\mathbb{H}^\ell \xrightarrow{\psi} X) < \mathbb{Z}$ & $G \simeq \pi_1(X)$. We have 2 cases to analyze:

(1) If $G = \{0\}$, then $X$ is simply connected so $f : X \longrightarrow \mathbb{D}^*$ is also the universal cover of $\mathbb{D}^*$. By uniqueness (up to iso), $\psi$ is biholomorphic.

(2) If $G = k\mathbb{Z}$ for $k \geq 1$, then by Theorem 2 §7.2 $X \simeq \mathbb{H}^\ell / G$ (*) & $\psi$ is the quotient map

Consider $g : \mathbb{H}^\ell \longrightarrow \mathbb{D}^*$
$g(z) \longrightarrow \exp(z/k)$

Then $g$ is the universal covering and

$g(z_1) = g(z_2) \Leftrightarrow z_1 \sim_G z_2$. (**)

$$\begin{tikzcd} & \mathbb{H}^\ell \arrow[dl, "\psi"'] \arrow[d, "g"] \arrow[dd, bend left=60, "\exp"] \\ X \arrow[r, "\exists \varphi"] \arrow[dr, "f"'] & \mathbb{D}^* \arrow[d, "p_k"] \\ & \mathbb{D}^* \end{tikzcd}$$

$\mathrm{Deck}(\mathbb{H}^\ell \mid \mathbb{D}^*) = G$ (The cover is Galois) by Theorem 1 §7.1

So $\exists\, \varphi : X \longrightarrow \mathbb{D}^*$
$x \longmapsto g(a)$ for $a$ with $\psi(a) = x$

& $\varphi \circ \psi = g$.

By construction $\quad p_k\circ\varphi(x) = p_k\circ g(a) = \exp(a) \quad$ if $\varphi(a) = x$
$$\left.\begin{array}{l} p_k\circ\varphi(x) = p_k\circ g(a) = \exp(a) \\ \quad = f\circ\varphi(a) = f(x) \end{array}\right\} \Rightarrow f = p_k\circ\varphi.$$

The proof of Thm 2 §7.2 shows $\varphi$ is a covering, so holomorphic. Now, $\varphi$ is a bijection by construction (use (*) & (**)), and it is a local biholomorphism. Thus, it is a biholomorphism & $k = N$ (# sheets of $f$)

<u>Theorem 2</u>: Fix $X$ a R.S & $f: X \to \mathbb{D}$ a proper non-constant holomorphic map which is unbranched over $\mathbb{D}^*$. Then $\exists\ \varphi: X \to \mathbb{D}$ biholomorphism st

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "f"'] & \mathbb{D} \arrow[d, "p_N"] \\ & \mathbb{D} \end{tikzcd} \quad \text{for some } N \geq 1.$$

<u>Proof</u>: Consider $X^* = X \setminus f^{-1}(\{0\}) = f^{-1}(\mathbb{D}^*)$ & $f|_{X^*}: X^* \to \mathbb{D}^*$.

Then $f|_{X^*}$ is a proper, discrete map & so it has finite fibers (by Thm 1 §5.2)

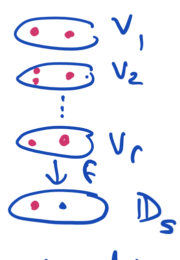
Since $f|_{X^*}$ is a proper, local homeomorphism & $\mathbb{D}^*$ is connected, we see that $f|_{X^*}$ is a covering (use Thm 2 §5.2) with finitely many sheets, say $N$.

We use Thm 1 §7.3 to build $\underset{\text{biholo}}{\varphi: X^* \to \mathbb{D}^*}$ filling

$$\begin{tikzcd} X^* \arrow[r, "\varphi"] \arrow[dr, "f|_{X^*}"'] & \mathbb{D} \arrow[d, "p_N"] \\ & \mathbb{D}^* \end{tikzcd}$$

• To extend $\varphi$ to $\varphi: X \to \mathbb{D}$, we declare $\varphi(b) = 0 \quad \forall\, b \in f^{-1}(0) \Rightarrow p_N\circ\varphi = f$

• So $\varphi$ is bounded on $X^*$, the extension is holomorphic. If we show $\varphi$ is bijective we get $\varphi$ is biholomorphic. This boils down to the following:




<u>Claim</u>: $|f^{-1}(0)| = 1$

Pf/ We use the fact that $f|_{X^*}$ is a covering

If $f^{-1}(0) = \{b_1, \ldots, b_r\}$, pick disjoint neighborhoods $V_1, \ldots, V_r$ of $b_1, \ldots, b_r$, resp. & $\mathbb{D}(s) \subseteq \mathbb{D}$ $(0 < s < 1)$ st $\boxed{f^{-1}(\mathbb{D}_s) \subseteq V_1 \sqcup \cdots \sqcup V_r}$ (***)

This is done by using the def of ramification index, the properness of $f$ & Thm 1 (2) §5.2

But $f^{-1}(\mathbb{D}_s^*) \underset{\varphi^*}{\simeq} \mathbb{D}^*(s^{1/n})$ which is connected

Note, each $b_j$ is an accumulation pt of $f^{-1}(\mathbb{D}_s^*)$, so $f^{-1}(\mathbb{D}_s) = \overline{f^{-1}(\mathbb{D}_s^*)}$ is also connected. By (***) we get $f^{-1}(\mathbb{D}_s) \subseteq V_j$ for a single $j$, which means $r=1$ (each $b_j \in f^{-1}(\mathbb{D}_s) \cap V_j$ by construction).

## §8.1. Basics on Sheaves:

Fix $X$ a topological space. & $\mathcal{C}$ = category of vector spaces / ab groups / rings, etc.

**Definition:** A presheaf on $X$ with values in $\mathcal{C}$ is a pair $(\mathcal{F}, \rho)$ where:

(1) $\mathcal{F}$ is an assignment : $U \subseteq X$ open $\longmapsto \underbrace{\mathcal{F}(U)}_{=\Gamma(U,\mathcal{F})} \in \mathrm{Obj}(\mathcal{C})$. (sections on $U$)

(2) For each pair $V \subseteq U \subseteq X$ of opens, we have $\mathcal{F}(U) \xrightarrow[\text{(restriction map)}]{\rho_{U,V}} \mathcal{F}(V) \in \mathrm{Homs}(\mathcal{C})$

satisfying the following properties:

(i) $\rho_{U,U} = id_{\mathcal{F}(U)}$ $\forall\, U \subseteq X$ open

(ii) For each triple $W \subseteq V \subseteq U$ of opens in $X$ we have

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\rho_{U,V}"] \arrow[rr, bend right, "\rho_{U,W}"'] & \mathcal{F}(V) \arrow[r, "\rho_{V,W}"] & \mathcal{F}(W) \end{tikzcd} \qquad \rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$$

We omit $\rho$ when understood from context and write $\boxed{f|_V := \rho_{U,V}(f)}$ for $f \in \mathcal{F}(U)$.

**Example** (1) $\mathcal{F}(U) = \{ f: U \to \mathbb{C}, f \text{ is cont} \}$ $\rho_{U,V}$ = usual restrictions.

⇝ presheaf of vector spaces

(2) $G$ = ab gp ⇝ $\mathcal{G}(U) = \{ f: U \to G \text{ constant} \}$ $\rho_{U,V}$ = usual restriction.

⇝ presheaf of abelian groups

**Definition:** A presheaf $(\mathcal{F}, \rho)$ is a sheaf if the following local gluing axiom holds (cocycle condition)

" For every open $U$ of $X$ & every open cover $\{U_i\}_{i \in I}$ of $U$ $(U = \bigcup_{i \in I} U_i)$ with $f_i \in \mathcal{F}(U_i)$ $\forall i$ satisfying $\rho_{U_i, U_i \cap U_j}(f_i) = \rho_{U_j, U_i \cap U_j}(f_j)$ $\forall i,j$

$(f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, ie agreement on the overlaps) we have a <u>unique</u> $f \in \mathcal{F}(U)$ with $\rho_{U,U_i}(f) = f_i \quad \forall i \in I$ "

<u>Examples</u> $X$ = RS.

(1) Presheaf of constant functions is <u>not</u> a sheaf ($U_1 \cap U_2 = \emptyset$ with 2 different constant functions $f_1 : U_1 \to \mathbb{C}$, $f_2 : U_2 \to \mathbb{C}$ cannot be glued to a constant function on $U_1 \cap U_2$). To get a sheaf we need to work with <u>locally constant</u> complex valued functions. Write it as $\tilde{\mathcal{G}}$.

- We have 2 sheaves of commutative algebras:

(2) $\mathcal{O}$ = sheaf of holomorphic functions ($\rho$ = usual restr.) $(= \mathcal{O}_X)$

$\mathcal{O}(U) = \{f : U \to \mathbb{C} \text{ holo function}\}$

(3) $\mathcal{M} = \mathcal{M}_X$ = sheaf of meromorphic functions on $X$; $\mathcal{M}(U) = \{f : U \to \mathbb{C} \text{ mero function}\}$

(4) $\mathcal{O}^*$ = sheaf of non-vanishing holomorphic functions

$\mathcal{O}^*(U) = \{f : U \to \mathbb{C}^* \text{ holo function}\}$.

This is a sheaf of (multiplicative) ab groups.

(5) $\mathcal{M}^*$ = sheaf of not locally vanishing meromorphic functions

$\mathcal{M}^*(U) = \{f : U \to \mathbb{C}$ mero function $\mid f$ is not vanishing identically on a connected component of $U\}$.

Again, this is a sheaf of mult. ab gps.

- There is a way to build a sheaf from a presheaf, called "<u>sheafification</u>". The idea is to formally add sections that will allow to glue the old sections on the presheaf. The construction involves direct limits. We will not need it in this course.

- Our next notion leads to considering germs of functions. It is the stalk of a presheaf at a point. It is defined via an equivalence relation.

Fix a presheaf $\mathcal{F}$ on a topological space $X$.

**Definition:** Given $x \in X$, we define the stalk of $\mathcal{F}$ at $x$ as a direct limit

$$\mathcal{F}_x = \varinjlim_{\substack{U \text{ open} \\ x \in U}} \mathcal{F}(U) = \bigsqcup_{x \in U \subseteq X} \mathcal{F}(U) / \sim_x$$

where $\sim$ is defined as follows: if $f_1 \in \mathcal{F}(U_1)$, $f_2 \in \mathcal{F}(U_2)$ we say $f_1 \sim_x f_2$ if $\exists\ V \underset{\text{open}}{\subseteq} U_1 \cap U_2$ with $x \in V$ st $f_1|_V = f_2|_V$ (This is an equivalence relation)

**Note:** The construction comes with a natural map

$$\begin{aligned} \mathcal{F}(U) &\xrightarrow{\rho_x} \mathcal{F}_x \\ f &\longmapsto \bar{f} = \text{equiv class of } f. \end{aligned}$$

It is a morphism in the category where $\mathcal{F}$ lies

**Examples:** Stalks for $\mathcal{O}_X$ & $\mathcal{M}_X$ on $X = RS$ = germs of holo / mero functions near a pt $x \in X$ $(\mathcal{O}_{X,x}, \mathcal{M}_{X,x})$

If $X = \mathbb{C}$: $\mathcal{O}_a \cong$ convergent power series in $(z-a)$ (with positive ROC)

$\mathcal{M}_a \cong$ convergent Laurent power series in $(z-a)$ (on an annulus with center $a$)

**Remark:** We have a natural evaluation map

$$\begin{aligned} \mathcal{O}_{X,x} &\longrightarrow \mathbb{C} \\ \bar{f} &\longmapsto f(x) \end{aligned}$$

We can test for triviality of a section on a sheaf of ab groups via stalks:

**Lemma 1:** Suppose $\mathcal{F}$ is a sheaf of sets on $X$. Pick $U \subseteq X$ open, then the map

$$\begin{aligned} \mathcal{F}(U) &\longrightarrow \prod_{x \in U} \mathcal{F}_x \quad \text{is injective.} \\ f &\longmapsto (\rho_x(f))_x \end{aligned}$$

Pf/ Use the sheaf axiom + $\rho_x(f) = \rho_x(g) \iff \exists\ x \in V \subseteq U$ with $f|_V = g|_V$ $\square$

## §8.2. The topological space of a presheaf:

Given a presheaf $\mathcal{F}$ on a top space $X$, we build a space $|\mathcal{F}|$ using the stalks & a natural map $p: |\mathcal{F}| \longrightarrow X$ (called the "projection map")

**Definition:** $\boxed{|\mathcal{F}|} = \bigsqcup_{x \in X} \mathcal{F}_x$ & $\boxed{p}: |\mathcal{F}| \longrightarrow X$ $\quad$ $(\eta \in \mathcal{F}_x \leftrightarrow (x, \eta))$

$$\eta \in \mathcal{F}_x \longmapsto x$$

with $p|_{\mathcal{F}_x} = \rho_x$

We endow $|\mathcal{F}|$ with a topology by fixing a basis:

<u>Definition:</u> Given $U \subseteq X$ open & $f \in \mathcal{F}(U)$ we set

$$\mathcal{N}(U,f) = \{ \rho_x(f) \in \mathcal{F}_x : x \in U \}$$

<u>Theorem 1:</u> $\mathcal{B} = \{ \mathcal{N}(U,f) : U \text{ open}, f \in \mathcal{F}(U) \}$ is a subbasis for a topology on $|\mathcal{F}|$

Furthermore, this topology makes $p : |\mathcal{F}| \longrightarrow X$ into a local homeomorphism.

<u>Proof:</u> • We need to verify the two axioms for a basis:

(1) $\eta \in \mathcal{F}_x \subseteq |\mathcal{F}|$ then pick $U$ open with $x \in U$ & $f \in \mathcal{F}(U)$ st $\rho_x(f) = \eta$. Then $\eta \in \mathcal{N}(U,f)$.

(2) $\eta \in \mathcal{N}(U_1, f_1) \cap \mathcal{N}(U_2, f_2)$ with $\eta \in \mathcal{F}_x$, so $x \in U_1 \cap U_2$ & $\rho_x(f_1) = \rho_x(f_2) = \eta$. Then $\exists\ x \in V \subseteq U_1 \cap U_2$ open & $g \in \mathcal{F}(V)$ with $f_1|_V = g = f_2|_V$. This means $\eta \in \mathcal{N}(V, g)$

Furthermore $\rho_y(g) = \rho_y(f_1) = \rho_y(f_2)$ $\forall y \in V$, so $\mathcal{N}(V,g) \subseteq \mathcal{N}(U_1, f_1) \cap \mathcal{N}(U_2, f_2)$

• To finish, we show $p$ is a local homeo. This follows from the assertion:

<u>Claim</u>: $\varphi = p|_{\mathcal{N}(U,f)} : \mathcal{N}(U,f) \longrightarrow U$ is a homeomorphism $\forall U$ open, $f \in \mathcal{F}(U)$

Pf/ • $\varphi$ is a bijection & $\varphi^{-1} : U \longrightarrow \mathcal{N}(U,f)$, $x \longmapsto \rho_x(f)$

• <u>$\varphi$ is open</u>: $\varphi(\mathcal{N}(V,g)) = V$ for $\mathcal{N}(V,g) \subseteq \mathcal{N}(U,f)$.

• <u>$\varphi$ is continuous</u>: If $V \subseteq U$ is open, then $\varphi^{-1}(V) = \{ \rho_x(f) \text{ for } x \in V \} = \mathcal{N}(V, f|_V)$ □

<u>Q</u>: How good is this topology?

<u>A</u> It depends on the topology of $X$ & how can we equate sections of $\mathcal{F}$ from their equiv class on <u>same</u> stalk. This last condition is known as the Identity Thm, which we have for sheaves $\mathcal{O}_X$ & $\mathcal{C}_X$ (Lemma 1 §8.1 is weaker than what we need).

**Definition**: A presheaf $\mathcal{F}$ on a top space $X$ satisfies the Identity Theorem if the following condition holds: $\forall\, Y \subseteq X$ open & connected we have: if $f, g \in \mathcal{F}(V)$ satisfy $\rho_a(f) = \rho_a(g)$ for some $a \in Y$, then $f = g$".

**Remark**: • This will hold for $\mathcal{O}$ & $\mathcal{M}$ sheaves on $X$ (by the Identity Thm §2.3)

• The identity Thm fails for $\mathcal{C}$ = sheaf of cont $\mathbb{C}$-valued functions on $X$.

Note that we are only requiring $\overline{f} = \overline{g}$ on some $\mathcal{F}_a$, not for all $a \in Y$.

**Theorem 2**: Assume $X$ is a Hausdorff, locally connected top. space (eg $X$ a RS) & $\mathcal{F}$ is a presheaf on $X$ satisfying the Identity Theorem. Then, the top. space $|\mathcal{F}|$ is Hausdorff.

**Proof**: Pick $\eta, \xi \in |\mathcal{F}|$ with $\eta \neq \xi$. We need to analyze two cases:

**CASE 1**: $\eta \in \mathcal{F}_x$ & $\xi \in \mathcal{F}_y$ with $x \neq y$.

The Hausdorff condition on $X$ gives $U_1, U_2$ opens separating $x$ & $y$.

Pick opens $V_1, V_2$ with $x \in V_1 \subseteq U_1$, $y \in V_2 \subseteq U_2$ & $f_1 \in \mathcal{F}(V_1)$, $f_2 \in \mathcal{F}(V_2)$ with $\rho_x(f_1) = \eta$, $\rho_x(f_2) = \xi$.

Then $\eta \in \mathcal{N}(V_1, f_1)$, $\xi \in \mathcal{N}(V_2, f_2)$ & $\mathcal{N}(V_1, f_1) \cap \mathcal{N}(V_2, f_2) = \emptyset$

**CASE 2**: $\eta, \xi \in \mathcal{F}_x$

By construction, $\exists$ opens $V_1, V_2$ with $x \in V_1 \cap V_2$ and $f_1 \in \mathcal{F}(V_1)$ with $f_2 \in \mathcal{F}(V_2)$
$\rho_x(f_1) = \eta$ & $\rho_x(f_2) = \xi$.

Since $X$ is locally connected we can pick $W$ open & connected with $x \in W \subseteq V_1 \cap V_2$

**Claim**: $\mathcal{N}(W, \rho_{V_1,W}(f_1)) \cap \mathcal{N}(W, \rho_{V_2,W}(f_2)) = \emptyset$

Pf/ If $\varphi \in \mathcal{F}_y$ lies in the intersection, then $\rho_y(\rho_{V_1,W}(f_1)) = \varphi = \rho_y(\rho_{V_2,W}(f_2))$

The Identity Theorem says $\rho_{V,W}(f_1) = \rho_{V_2,W}(f_2)$. This contradicts $\eta \neq \xi$

since $\rho_x(\rho_{V,W}(f_1)) = \rho_x(f_1)$ & $\rho_x(\rho_{V_2,W}(f_2)) = \rho_x(f_2)$ □

<u>Corollary</u>: Given a R.S $X$, the Top space $|\mathcal{O}|$ is Hausdorff. Furthermore, using the local homeo $|\mathcal{O}| \xrightarrow{p} X$, we can make each connected component $Z$ of $|\mathcal{O}|$ into a RS by pulling back the structure of $X$ via $p|_Z$. And $p|_Z : Z \longrightarrow X$ becomes holo.

Next, we'll discuss how to build the connected comp of $|\mathcal{O}|$. For this, we need analytic cont. We'll cover this next time!

## §8.3 Aside on terminology:

Given a Top space $X$ & a presheaf $\mathcal{F}$ on $X$, it is usual to refer to elements of each $\mathcal{F}(U)$ ($U \subseteq X$ open) as "<u>sections of $\mathcal{F}$ on $U$</u>". The choice is made by the following fact:

By construction, a section to $p : |\mathcal{F}| \twoheadrightarrow X$ on $U$ is given by $s : U \longrightarrow |\mathcal{F}|$ with

$$\begin{tikzcd} & {|\mathcal{F}|} \arrow[d, "p"] \\ U \arrow[ur, dashed, "s"] \arrow[r, hook] & X \end{tikzcd}$$

$p \circ s = inc_U$.

So $\mathcal{F}(U) \xrightarrow{\Phi} \{ U \xrightarrow{s} |\mathcal{F}| \text{ continuous sections to } p \} := \mathcal{G}(U)$

$f \longmapsto (s(x) = \rho_x(f) \quad \forall x \in U)$

($s^{-1}(\mathcal{N}(V,g)) = V \cap U$ by construction so $s$ is automatically continuous.)

- The RHS defines a presheaf with usual restriction. Moreover, it is a sheaf

<u>Cocycle condition</u>: If $U = \bigcup_{i\in I} U_i$ & $U_i \xrightarrow{s_i} |\mathcal{F}|$ with $p \circ s_i = inc_{U_i}$ satisfy

$s_i|_{U_i \cap U_j} = s_j|_{U_i\cap U_j} \Rightarrow U \xrightarrow{s} |\mathcal{F}|$ with $s|_{U_i} = s_i$ is well-def & $p \circ s = inc_U$

- If $\mathcal{F}$ is a sheaf, then $\Phi$ is a bijection. Indeed, it's injective by Lemma §8.2 & since $p|_{\mathcal{N}(V,f)} : \mathcal{N}(V,f) \xrightarrow{\sim} V$ is homeo, we see that every cont section $s : U \longrightarrow |\mathcal{F}|$ there exist $\{(V_i, f_i)\}_{i\in I}$ with $\operatorname{im} s \subseteq \bigcup_{i \in I} \mathcal{N}(V_i, f_i)$, $s|_{V_i} = (p|_{\mathcal{N}(V_i,f_i)})^{-1}$ & $f_i|_{V_i\cap V_j} = f_j|_{V_i \cap V_j}$

$U = \bigcup_{i\in I} V_i$

($\rho_x(f_i) = \rho_x(f_j)$ $\forall x \in V_i \cap V_j$ so $f_i$ & $f_j$ agree locally around each $x \in V_i \cap V_j$ & $\mathcal{F}$ is a sheaf). The sheaf axiom allows us to glue $f_i$'s to $f \in \mathcal{F}(U)$. It's easy to check that $\Phi(f) = s$.