Lecture IX: Analytic continuation

Recall: X top space, $\mathcal{F}$ presheaf on X $\rightsquigarrow$ $|\mathcal{F}| = \bigsqcup_{x\in X} \mathcal{F}_x \xrightarrow{\ p\ } X$ $\quad p|_{\mathcal{F}_x} \equiv x$

$|\mathcal{F}|$ has a basis for its topology $\mathcal{N}(U,f) = \{\rho_x(f) \ \ f\in\mathcal{F}(U),\ x\in X\}$

THM 1: $|\mathcal{F}| \xrightarrow{\ p\ } X$ is local homeomorphism

THM 2: If $\mathcal{F}$ satisfies the Identity Theorem (for each U open connected & $f,g\in\mathcal{F}(U)$ we have $f=g$ whenever $\rho_a(f)=\rho_a(g)$ for some a) & X is Hausdorff & loc. connected then $|\mathcal{F}|$ is Hausdorff.

Corollary: For every X RS, $|\mathcal{O}|$ is Hausdorff & $p: |\mathcal{O}| \longrightarrow X$ is holomorphic.

$\Rightarrow$ We can make each connected component Z of $|\mathcal{O}|$ into a RS by pulling back the structure of X via $p|_Z$. Then $p|_Z : Z \longrightarrow X$ becomes holomorphic

Q: How to see if $|\mathcal{O}|$ is connected / determine its connected components?

A: Analytic continuation!

§9.1 Analytic continuation of germs along curves:

Fix a Riemann surface X and $\mathcal{O}=\mathcal{O}_X$ = sheaf of holomorphic functions on X.

Next goal: Study analytic continuations of germs of functions $f\in\mathcal{O}(U)$ along paths in X starting at pts in U. We work with germs of analytic functions

Definition: Given $a, b\in X$, φ in $\mathcal{O}_a$ & $u:[0,1]\longrightarrow X$ with $u(0)=a$, $u(1)=b$, the analytic continuation of φ along u (if it exists) is the unique lift

$\boxed{AC_u(\varphi)} = \hat{u} : [0,1] \longrightarrow |\mathcal{O}|$ of u relative to p with $\hat{u}(0)=\varphi$.

$$\begin{tikzcd} & {|\mathcal{O}|\ni\varphi} \arrow[d, "p"] \\ 0\in[0,1] \arrow[ur, "\hat{u}"] \arrow[r, "u"'] & X \end{tikzcd}$$

Remark: Uniqueness follows from I connected & p local homeomorphism.

If this exists, we say $\eta = AC_u(\varphi)(1) = \hat{u}(1) \in \mathcal{O}_b$ is the result of performing the analytic continuation of φ along $\hat{u}$.

<u>Equivalent Definition</u>: We need a continuous family $\varphi_t = \hat{u}(t) \in \mathcal{O}_{u(t)}$ for $t \in [0,1]$.

This means that for all $t_0 \in [0,1]$ we need to find

(1) an interval $T \subseteq [0,1]$ with $t_0 \in T$

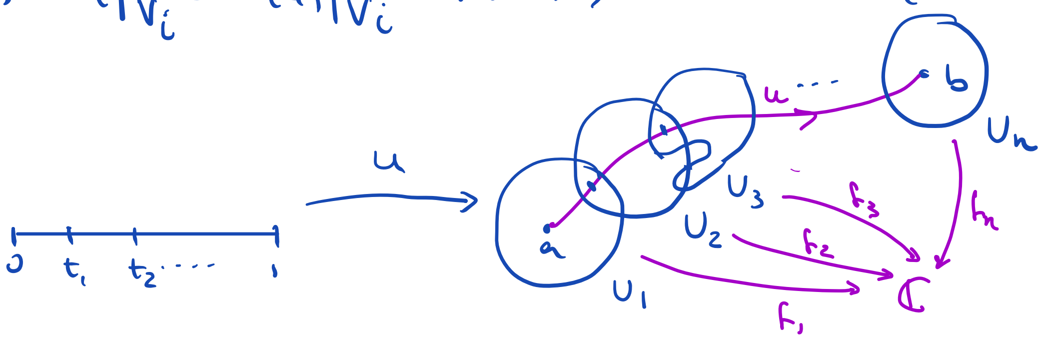
(2) an open $U_T \subset X$ with $u(T) \subseteq U_T$

(3) a function $f \in \mathcal{O}(U_T)$ with $\rho_{u(t)}(f) = \varphi_t \quad \forall t \in T$.

Since $[0,1]$ is compact we can use the Lebesgue number of a finite subcover of $\{T = T_{t_0}\}_{t_0}$ to get a partition $0 = t_0 < t_1 < \cdots < t_n = 1$, opens $U_1, \ldots, U_n$ of X with $u([t_{i-1}, t_i]) \subseteq U_i$ & $f_i \in \mathcal{O}(U_i)$ for $i = 1, \ldots, n$ with

(1) $\rho_a(f_0) = \varphi$, $\rho_b(f_{n-1}) = \eta$

(2) $f_i|_{V_i} = f_{i+1}|_{V_i} \quad \forall i = 1, \ldots, n-1$ where V_i is the! conn. comp. of $U_i \cap U_{i+1}$ containing $u(t_i)$.



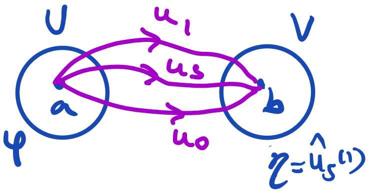
<u>Note</u>: Topology on $|\mathcal{O}|$ makes $\hat{u}(t) = \varphi_t$ into a continuous map.

<u>Q</u>: How much does $\eta = \hat{u}(1)$ depend on the choice of the path u?

<u>A</u>: Only depends on $[u] \in \pi_1(a,b)$ if we can lift homotopies.

<u>Monodromy Theorem</u>: Fix a Riemann surface X, two points $a, b \in X$ & two homotopic curves $u_0, u_1 : [0,1] \longrightarrow X$ joining a to b. Consider a homotopy $H : [0,1] \times [0,1] \longrightarrow X$ between u_0 & u_1 & set $u_s := H(-, s)$ for $0 \leq s \leq 1$. Assume the germ $\varphi \in \mathcal{O}_a$ admits an analytic continuation along each u_s. Then, the analytic continuations of φ along u_0 & u_1 give the same germ η in $\mathcal{O}_b$.

Pf/ This is a direct consequence of the Lifting Homotopies rel to p statement (see Theorem 1 §4.4). Since $|\mathcal{O}|$ & X are Hausdorff & p is a local homeomorphism.



$\square$

Remark: "Monodromy": take $a=b$ & u_0 null-homotopic path. $u_1 = \mathbb{1}_a$

Take $u_0 = \mathbb{1}_a$ & $u = u_1 \sim u_0$. , we get $AC_u(\varphi) = AC_{u_0}(\varphi) = \varphi$

Corollary: If X is a simply connected R.S, $a \in X$ & $\varphi \in \mathcal{O}_a$ is a germ admitting an analytic continuation along any curve starting at a, then we can find a global holomorphic function $f \in \mathcal{O}(X)$ with $\rho_a(f) = \varphi$.

Pf/ Given $x \in X$, set $\psi_x \in \mathcal{O}_x$ be the function germ obtained from the analytic cont of φ along a path u joining a to x. The construction of ψ_x is path independent by the Monodromy Theorem. Set $f: X \longrightarrow \mathbb{C}$ $\quad f(x) = \psi_x$.

Then f is holomorphic on X (locally holomorphic) & $\rho_x(f) = \psi_x \quad \forall x$.

In particular, $\rho_a(f) = \varphi$.

By the Identity Theorem, f is uniquely determined. $\square$

- The dependency on the path will lead to a multivalued function (analytic cont.)

$\mathcal{O}_a \ni \varphi \underset{u}{\rightsquigarrow} \hat{u}_{(1)} \in \mathcal{O}_b$ $\qquad$ We'll make this precise next time

The formal construction of AC will achieve 2 things:

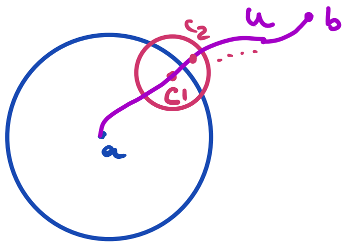
1. solve the multivaluedness issue
2. Find a holomorphic function on a RS. X "representing" the germ φ. This will allow us find all germs that can be constructed as AC of φ.

§9.2. Weierstrass "analytic function element"

- The original construction of analytic cont. is due to Weierstrass for $X = \mathbb{C}$.

  $\varphi \in \mathcal{O}_a \longleftrightarrow$ convergent power series in $z-a$ in a disc $D = D_r(a)$ $r = \mathrm{ROC} > 0$.

Q: Can we extend φ beyond D?



Pick $c \in D$ & write down the power series expansion of φ near c & compute its ROC r_1. If $D_{r_1}(c) \not\subset \overline{D}$, then we've extended φ outside D & we can repeat the process with $c_2 \in D_{r_1}(c) \setminus \overline{D}$

Examples: ① $\varphi = \sum_{n \geq 0} z^n \in \mathcal{O}_0$ lifts to $\mathbb{D}$. We can extend it to $f = \frac{1}{1-z}$ on $\mathbb{C} \setminus \{1\}$

② $\varphi = \sum_{n \geq 0} z^{n!} \in \mathcal{O}_0$ gives a function on $\mathbb{D}$. This function cannot be extended beyond S^1 by power series centered at pts in $\mathbb{D}$ because the series diverges for any n^{th} root of 1. for $n \geq 1$ $(\Rightarrow r_1 < 1 - |c|)$.

Proposition: $p: |\mathcal{O}| \longrightarrow X$ is not a covering map

Pf/ It's enough to find a curve on X with no lifting relative to p.
Since X is a RS, we can replace X by an open chart $U \xrightarrow{\sim} \mathbb{D}_2(0) \subseteq \mathbb{C}$ $(\mathbb{D} = \mathbb{D}_1(0))$
Take $f = \sum_{n \geq 0} z^{n!}$ in $\mathcal{O}(\mathbb{D})$ & consider the germ $\varphi = \rho_0(f) \in \mathcal{O}_0$.

Pick $u: [0,1] \longrightarrow X$ $u(t) = 2t$ Since f diverges on a dense set of S^1 (namely, $\exp(2\pi i \mathbb{Q})$) we see that f does not have an analytic continuation along u with initial value φ. In particular,

$$\begin{tikzcd} & {|\mathcal{O}|} \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "\nexists \hat{u}"] \arrow[r, "u"'] & X \end{tikzcd}$$

$\hat{u}(0) = \varphi$ $\square$

Lemma: $|\mathcal{O}|$ is locally pathwise connected BUT not connected

Pf/ • Since X is locally pathwise conn. & p is a local homeo, the same holds for $|\mathcal{O}|$

• If $|\mathcal{O}|$ were connected, then we would have $|\mathcal{O}|$ path connected, so any germ would be connected to another by a path in $|\mathcal{O}|$. This would say any 2 germs of analytic functions would be analytic cont. of each other, which we know is false $\square$

• Next, we describe <u>Weierstrass "analytic function element"</u>.

$$\mathcal{S} := \{ (f, D_r(a)) : f : D_r(a) \longrightarrow \mathbb{C} \text{ holo} \}$$

so f has a power series expansion in $(z-a)$ & roc $\geq r$.

We define an equivalence relation on $\mathcal{S}$.

$$(f, D_r(a)) \sim (g, D_s(b)) \quad \text{if } \exists\, u : [0,1] \longrightarrow \mathbb{C} \text{ with } \begin{matrix} u(0)=a \\ u(1)=b \end{matrix}$$

st the analytic conts of $\rho_a(f)$ along u exists & agrees with g. More precisely

$$AC_u(f)_{(1)} = \rho_b(g)$$

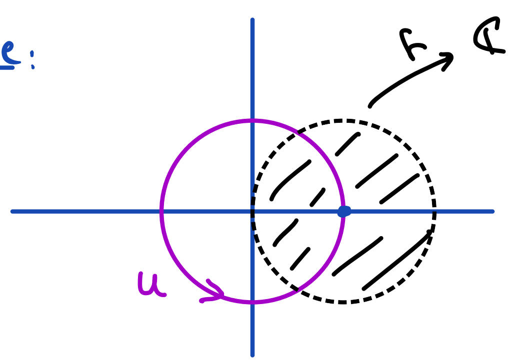
<u>Definition</u>: A <u>function element</u> is an equivalence class of $\mathcal{S}$ under this relation.

<u>Remark</u>: Since local lifting rel to p exist & paths in $D_r(a)$ are homotopic, we get

$$(f, D_r(a)) \sim (f, D_{r-|b-a|}(b)).$$

More generally, small perturbations of u give the same analytic function.

<u>Example</u>:



$$f(z) = z^{1/2} = (1-(1-z))^{1/2} = \sum_{n=0}^{\infty} \frac{(1-z)^n}{n!} \frac{1}{2}\left(\frac{1}{2}-1\right) \cdots \left(\frac{1}{2}-n+1\right)$$

If we go around u, we get $-z^{1/2} = \hat{u}_{(1)}$

So $(f, D_1(1)) \sim (-f, D_1(1))$

(More details next time!)

Conclusion: Given $\Psi \in \mathcal{O}_a$, & $(f, D_r(a)) \in S$ with $\rho_a(f) = \Psi$, the Weierstrass function element corresponding to $(f, D_r(a))$ is exactly the connected comp Z of $|\mathcal{O}|$ containing Ψ. The result will be a R.S. since $p|_Z : Z \longrightarrow X$ will be a local homeomorphism.

§9.3 Analytic continuation (formal construction)

Fix Y a RS & $\mathcal{O} = \mathcal{O}_Y$ sheaf of holomorphic $\mathbb{C}$-valued functions

Next, we formalize Weierstrass construction: analytic continuation is a multi-valued function, since it depends on the path u starting at $a \in Y$. In order to consider the space of all possible analytic extensions of $\varphi \in \mathcal{O}_a$, we'll work with the connected comp of $|\mathcal{O}|$ containing φ. This will give us a natural Riemann surface $Y \ni \Psi$ + a local homeo

$$p: \begin{array}{ccc} X & \longrightarrow & Y \\ \xi & \longmapsto & y \end{array} \qquad \text{if } \xi \in \mathcal{O}_y.$$

Definition 1: (Pullback & Pushforward of germs)

Fix $f: X \longrightarrow Y$ local homeomorphism between R.S

- Then, we get a pull back map of germs $f^*: \mathcal{O}_{Y, f(x)} \longrightarrow \mathcal{O}_{X,x}$ $\quad f^*(\eta) = \rho_x(g \circ f)$ if η is represented around $y = f(x)$ by $g \in \mathcal{O}_Y(V)$ with $y \in V$ open.

(This is well-defined & an isomorphism because f is a local homeomorphism)

Recall: Given a presheaf $\mathcal{F}$ on X, we have $f_*(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$ $\quad \forall U \subseteq Y$ open.

$\Rightarrow$ $f_*(\mathcal{F})$ is a presheaf on Y.

This descends to a map on stalks $f_*: \mathcal{F}_x \longrightarrow f_*(\mathcal{F})_{(f(x))}$ where if $s \in \mathcal{F}(V)$ & $f(x) \in V$ open, then $f_*([s]) = [s|_{f^{-1}(U) \cap V}]$

In our case, we get the push-forward map $f_*: \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Y, f(y)}$

Remark: Since f is a local homeo, we get $p_* = (p^*)^{-1} : \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Y,f(x)}$

Definition 2: Fix Y a RS, $a \in Y$ & $\varphi \in \mathcal{O}_a$. An <u>analytic continuation</u> of φ is a quadruple (X, p, f, b) where:

(1) X is a RS & $p : X \longrightarrow Y$ is an unbranched holomorphic map

(2) $f : Y \longrightarrow \mathbb{C}$ is a holomorphic function on Y.

(3) b is a pt in X with $p(b) = a$ & $p_*(\rho_b(f)) = \varphi \in \mathcal{O}_a$.

• We say (X, p, f, b) is <u>maximal</u> (or universal) if it satisfies the following property: "Given (Z, q, g, c) another analytic cont., there exists $F : Z \longrightarrow Y$ holo satisfying $p \circ F = q$ ("fiber preserving") with $F(c) = b$ & $f \circ F = g$

$$\begin{tikzcd} Z \arrow[r, "F"] \arrow[dr, "q"'] & Y \arrow[d, "p"] \arrow[dl, phantom, "\circlearrowleft"] \\ & X \end{tikzcd} \quad \& \quad \begin{tikzcd} Z \arrow[r, "F"] \arrow[dr, "g"'] & Y \arrow[d, "f"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathbb{C} \end{tikzcd}$$

Lemma 1: A maximal analytic continuation of $\varphi \in \mathcal{O}_a$ is unique up to isomorphism

Pf/ Take two mxl analytic continuations (X, p, f, b) & (Z, q, g, c) so $\exists\, \underset{(X \text{ mxl})}{F : Z \longrightarrow X}$ & $\underset{(Z \text{ mxl})}{G : X \longrightarrow Z}$ holomorphisms with $F(c) = b$, $G(b) = c$ &

$$\begin{tikzcd} Z \arrow[r, "F"] \arrow[dr, "q"'] & X \arrow[d, "p"] \arrow[dl, phantom, "\circlearrowleft"] \\ & Y \end{tikzcd}, \quad \begin{tikzcd} Z \arrow[dr, "q"'] & X \arrow[l, "G"'] \arrow[d, "p"] \arrow[dl, phantom, "\circlearrowleft"] \\ & Y \end{tikzcd} \quad \& \quad \begin{tikzcd} Z \arrow[r, "F"] \arrow[dr, "g"'] & X \arrow[d, "f"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathbb{C} \end{tikzcd}, \quad \begin{tikzcd} Z \arrow[dr, "g"'] & X \arrow[l, "G"'] \arrow[d, "f"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathbb{C} \end{tikzcd}$$

But then $G \circ F : Z \longrightarrow Z$ satisfies

$$\begin{tikzcd} & Z \arrow[d, "q"] \\ Z \arrow[ur, "G \circ F"] \arrow[r, "q"'] \arrow[ur, phantom, "\circlearrowleft", shift right=3] & Y \end{tikzcd} \qquad G \circ F(b) = b$$

So $G \circ F$ is a lifting of q rel. to q with $G \circ F(b) = b$. Since Z is connected, Theorem 2 §4.3 ensures $G \circ F = id_Z$ (liftings are unique!)

Same reasoning gives $F \circ G = id_Y$. We conclude that F is a biholomorphism.

$\Rightarrow$ $q = p \circ F$ is uniquely determined

- $g = f \circ F$ ____________
- $c = F^{-1}(b)$ ____________

$\square$

Q: How to build a maximal analytic continuation?

A Use connected comp X of $|\mathcal{O}|$ containing $\varphi \in \mathcal{O}_{Y,a}$

Theorem: Given a RS Y & $\varphi \in \mathcal{O}_{Y,a}$ for some $a \in Y$, the exists a mxl analytic continuation. Moreover, it equals (X, p, f, φ), where

(1) X is the connected comp of $|\mathcal{O}_Y|$ containing $\varphi \in \mathcal{O}_{Y,a}$ &

(2) $p = p|_X : X \longrightarrow Y$, $\eta \longmapsto y$ if $\eta \in \mathcal{O}_{Y,y}$.

(3) $f: X \longrightarrow \mathbb{C}$ $\quad f(\eta) = \eta(p(\eta))$

Proof: By construction, X is a RS & p is holomorphic. In addition, $p(\varphi) = a$
To show that (X, p, f, φ) is an analytic cont of φ, it remains to check 2 things.
We do this in 2 separate claims.

Claim 1: f is holomorphic

Pf/ Recall p is a local homeomorphism $p: \mathcal{N}(U,s) \xrightarrow{\sim} U$ for $U \subseteq Y$ open conn $s \in \mathcal{O}_Y(U)$

given $\xi \in X \Rightarrow \exists\ U \subseteq Y$ & $s \in \mathcal{O}_Y(U)$ with $\xi \in \mathcal{N}(U,s)$.

- U is connected, so $\mathcal{N}(U,s)$ is also connected. Since X is the con comp of ξ, we get $\xi \in \mathcal{N}(U,s) \subseteq X$

$\Rightarrow f|_{\mathcal{N}(U,s)}([s]_y) = [s]_y(p([s]_y)) = s(y) \quad \forall y \in U \qquad ([s]_y \in \mathcal{O}_{Y,y})$

$$\Rightarrow \begin{tikzcd} \xi \in \mathcal{N}(U,s) \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{C} \\ U \arrow[ur, "s \text{ holo}"'] & \end{tikzcd}$$

$\Rightarrow f$ is holomorphic by definition.

Claim 2: $p_*(\rho_\varphi(f)) = \varphi$

Pf/ We pick $\mathcal{N}(U,s)$ with $\varphi \in \mathcal{N}(U,s)$ $\quad (a \in U \;\&\; [s]_a = \varphi)$

$$\Longrightarrow \quad p_*(\underbrace{\rho_\varphi(f)}_{\in \mathcal{O}_{X,\varphi}}) = \rho_{p(\varphi)}(s) = [s]_a = \varphi$$

(This is more general: $\forall \xi \in \mathcal{N}(U,s)$ & $y \in U$ with $\xi \in \mathcal{O}_{Y,y}$ we get $p_*\rho_\xi(f)) = \rho_y(s) = \xi$)

• To finish, we need to show the quadruple is a maximal A.C. Say we are given another AC (Z, q, g, c). We want to define $F: Z \longrightarrow X$ holo with $p \circ F = q$, $F(c) = b$ & $f \circ F = g$

Pick $z \in Z$ & $y = q(z)$. Since Z is path connected (it is a RS) we can find a path $v: [0,1] \to Z$ joining c to z.

Then $u = q \circ v : [0,1] \longrightarrow Y$ is a path joining a to $q(z) = y$.

Claim 3: $\eta := q_*(\rho_z(g)) = AC_u(\varphi)_{(1)}$ $\quad$ (see Lemma 2 below)

In particular, $\eta \in X$ (paths in $|U|$ starting from φ correspond to analytic continuations of φ by Definition §9.1). We set $\boxed{F(z) := \eta}$.

Claim 4: F is well-def, holomorphic, $p \circ F = q$, $F(c) = \varphi$ & $f \circ F = g$.

Pf/ • $q_*(\rho_z(g))$ is indep of v $\Rightarrow$ F is well-def.

• $p \circ F_{(z)} = p(\underbrace{AC_{q\circ v}(\varphi)_{(1)}}_{\in \mathcal{O}_{Y,q(z)}}) = q_{(z)}$ by def of p.

• Since q & p are local biholomorphisms, $p \circ F = q$ forces F to be holomorphic

• If $z = c$ we take $v = \mathbb{1}_b$ $\Rightarrow$ $F(c) = AC_u(\varphi)_{(1)} = \varphi$ (by the uniqueness of lifts relative to q, $\hat{u} = v$)

• $f \circ F_{(z)} = f(AC_{f\circ v}(\varphi)_{(1)}) = f(\underbrace{q_*(\rho_z(g))}_{=[s]_y}) = [s]_y = AC_{f\circ v}(\varphi)_{(1)}\Big|_y$

$F(z) \in \mathcal{N}(U,s)$ $\quad s \in \mathcal{O}_Y(U)$ $\quad y \in U$

$$= \left. q_*(\rho_z(g))(1) \right|_{q(z)} \underset{\substack{\downarrow \\ q \text{ local homeo}}}{=} \rho_z(g)(z) = g(z).$$

$$\left(q_* \ \mathcal{O}_{Z,z} \longrightarrow \mathcal{O}_{Y,q(z)} \quad [h] \longmapsto [h \circ q^{-1}] \right)$$
$h \in \mathcal{O}(U)$
$q : U \xrightarrow{\sim} V$ homeo, $x \mapsto q(x)$

$\square$

To prove Claim 3 we use the following lemma:

Lemma 2: Fix a R.S. Y, $a \in Y$, $\varphi \in \mathcal{O}_a$ & an analytic continuation (Z, q, g, c) of φ. Fix $v : [0,1] \longrightarrow Z$ a curve with $v(0) = c$ & write $z := v(1)$. Then, the function germ $\psi = q_* \rho_z(g) \in \mathcal{O}_{Y, q(z)}$ is an analytic continuation of φ along the curve $u = q \circ v : [0,1] \longrightarrow X$ with starting point a.

Proof: Given $t \in [0,1]$ we write $\varphi_t = q_*(\rho_{v(t)}(g)) \in \mathcal{O}_{Y, q\circ v(t)}$. Then:

(1) $\varphi_0 = q_*(\rho_c(g)) = \varphi$ (by Definition 2 §9.4)

(2) $\varphi_1 = q_*(\rho_z(g)) = \psi$ (by def of ψ)

To show $\psi = AC_u(\varphi)(1)$ we use that Z is connected & q is a local homeomorphism, so $q_* : \mathcal{O}_{Z,z'} \longrightarrow \mathcal{O}_{Y, q(z')}$ is an isomorphism as well $\forall z' \in Z$

Take $u = q \circ v : [0,1] \longrightarrow X$ $\quad u(0) = a$, $u(1) = q(z)$

$$\begin{tikzcd} & \lvert \mathcal{O} \rvert \ni \varphi \arrow[d, "p"] \\ a \in [0,1] \arrow[r, "u"'] \arrow[ur, "?\exists \hat{u}"] & Y \end{tikzcd}$$

We want to build $\hat{u} : [0,1] \longrightarrow Z$ lifting of u rel to p with $\hat{u}(0) = c$.

Claim: $\hat{u}(t) := \varphi_t$ for all t.

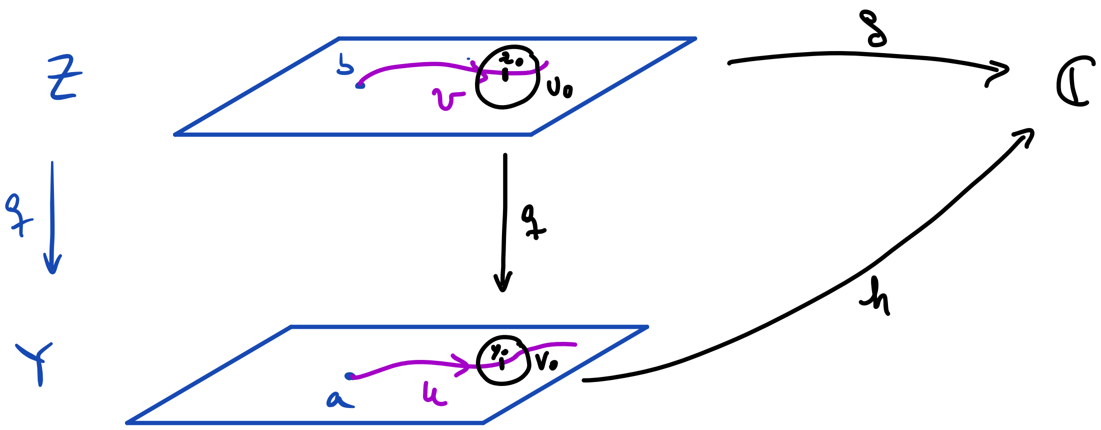
Pf/. • $p(\varphi_t) = p(q_*(\rho_{v(t)}(g))) = q \circ v(t) = u(t)$ $\forall t$. ; $\varphi_0 = \varphi$ ✓

We check that φ_t satisfies the equiv characterization of $AC_u(\varphi)$.

We use the global function $g : Z \longrightarrow \mathbb{C}$ to build patches for $AC_u(\varphi)$ along opens containing $u_{[0,1]}$. These patches will glue because g exists.

For each $t_0 \in [0,1]$, write: $y_0 = u(t_0) = q \circ v(t_0) = q(z_0)$ Since q is a local homeo $\exists z_0 \in U_0 \subseteq Z$ & $y_0 \subseteq V_0 \subseteq Y$ opens with $q|_{U_0} : U_0 \xrightarrow{\sim} V_0$ homeo.

We write $h = g \circ (q|_{U_0})^{-1} : V_0 \longrightarrow \mathbb{C}$, so $h \in \mathcal{O}_Y(V_0)$



Then $g = q^*(h) \in \mathcal{O}_Z(U_0)$ & we have $q_*(\rho_\eta(g)) = \rho_{q(\eta)}(h)$. $\forall \eta \in U_0$

Pick open nbhd T of t_0 in $[0,1]$ with $v(T) \subseteq U_0$. For each $t \in T$ we have

$$\rho_{u(t)}(h) = \rho_{q\circ v(t)}(h) = q_*(\rho_{v(t)}(g)) = \varphi_t . \qquad (\text{take } \eta = u(t))$$

These guys glue together (as in the spirit of Weierstrass's definition) so we get that φ_t is the unique lifting of u rel to p with $\hat{u}(0) = \varphi$.

<u>Conclude</u> $\hat{u}(1) = AC_u(\varphi)(1) = \varphi_1 = \psi$. $\square$