# Lecture X: Analytic continuation II & Elem symmetric functions

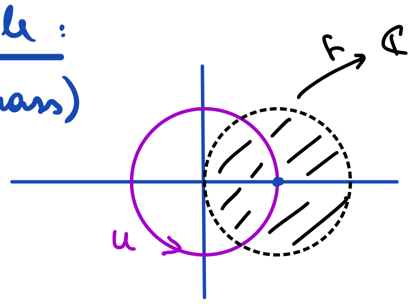
**Recall.** $Y$ RS, $\varphi \in \mathcal{O}_a$

① Given $u : [0,1] \to Y$, $u(0) = a$, we define $AC_u(\varphi)$ (if it exists) as $\hat{u}(1)$

where
$$\begin{tikzcd} & {|\mathcal{O}| \ni \varphi} \arrow[d, "p"] \\ {0 \in [0,1]} \arrow[ur, "\hat{u}"] \arrow[r, "u"'] & {Y \ni a} \end{tikzcd}$$
$\circlearrowleft$ & $\hat{u}(0) = \varphi$

Dependent of path unless we can lift homotopies relative to $p$.

**Example:** (Weierstrass)




$$f(z) = z^{1/2} = (1 - (1-z))^{1/2} = \sum_{n=0}^{\infty} \frac{(1-z)^n}{n!} \frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)$$

If we go around $u$, we get $-z^{1/2} = \hat{u}(1)$

$AC_u(z^{1/2}) = -z^{1/2}$.

② $|\mathcal{O}|$ is not connected

$\Rightarrow$ • **AC**: quadruple $(X, p, f, b)$
(1) $X$ RS $\quad p : X \to Y$ unbranched holomorphic
(2) $f : X \to \mathbb{C}$ holo
(3) $b \in X$, $p(b) = a$ & $p_*(\rho_b f) = \varphi$

$[\, p_* \; \mathcal{O}_{X,x} \to \mathcal{O}_{Y,p(x)} \qquad [s] \mapsto [s \circ p^{-1}] \qquad p : U \xrightarrow{\sim} V \quad x \in U, \; p(x) \in V \,]$

$AC(\varphi) \leftrightarrow$ path on $X$ with starting pt $\varphi$

• **MAC**: for any other AC quadruple $(Z, q, g, c)$ we have $F : Z \to X$ holo

with
$$\begin{tikzcd} Z \arrow[rr, "F"] \arrow[dr, "q"'] & {\circlearrowleft} & X \arrow[dl, "p"] \\ & Y & \end{tikzcd} \quad ; \quad \begin{tikzcd} Z \arrow[rr, "F"] \arrow[dr, "g"'] & {\circlearrowleft} & X \arrow[dl, "f"] \\ & Y & \end{tikzcd} \quad ; \quad F(c) = b$$

**THM** MAC of $\varphi$ exist.
$$\begin{cases} X = \text{connected component of } |\mathcal{O}| \text{ containing } \varphi \\ p = p|_X \text{ with } p : |\mathcal{O}| \to Y, \; \eta \mapsto y \quad \text{if } \eta \in \mathcal{O}_y. \\ f : X \to \mathbb{C} \qquad f(\eta) = \eta(y) \quad \text{if } \eta \in \mathcal{O}_y \quad (\text{This is defined on } |\mathcal{O}|) \\ b = \varphi \end{cases}$$

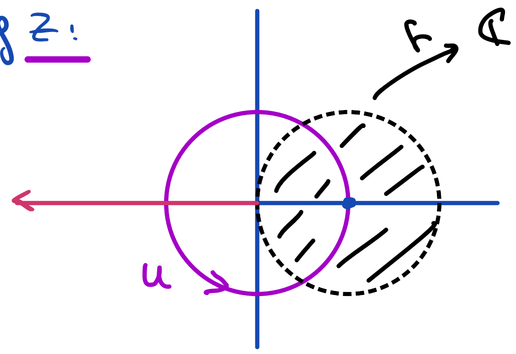
**Think:** $X$ = germs that can be obtained as analytic continuations of $\varphi$

= collection of all paths $u$ in $X$ with $u(0) = a$ along which $\varphi$ can be analytically cont $/\sim$
$(u \sim u'$ if $AC_u(\varphi) = AC_{u'}(\varphi))$

**Remark:** By using $|\mathcal{M}| \xrightarrow{p} Y$ we can do AC of meromorphic germs. $p$ will have branch points.

## §10.1 Examples:

① **$\log z$:**




$Y = \mathbb{C}^\times$, $a = 1$, $\varphi$ = series expn of $\log z$ near 1.

$$\varphi_{(z)} = \log(z) = \ln(1-(1-z)) = -\sum_{k=0}^{\infty} \frac{(1-z)^k}{k!}$$

We claim there exists a mxl analytic cnt of $\varphi$.

We take a cut $U = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ ( = cut for standard log)

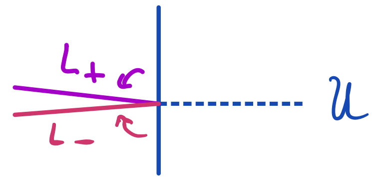
$$\Rightarrow \text{"}\log_0\text{"}(z) : U \longrightarrow \mathbb{C}$$
$$\ln(|z|) + i \arg(z) \quad \text{with} \quad \arg(z) \in (-\pi, \pi)$$

Similarly, we can define "$\log_k$" $: U \longrightarrow \mathbb{C}$ $\quad \log_k(z) = \log_0(z) + 2\pi i k$

- Now, take a path loop $u$ from 1 to 1 surrounding 0. $\Rightarrow AC_u : \mathcal{O}_1 \longrightarrow \mathcal{O}_1$

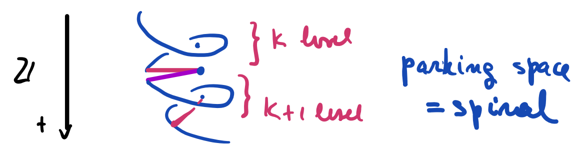
$$\Rightarrow AC_u(\log_k(z)) = \log_{k+1}(z)$$

$$Z = \underbrace{(\mathbb{C} \setminus \mathbb{R}_{\leq 0})}_{= U} \sqcup \underbrace{L_+ \sqcup L_-}_{2 \text{ copies of } \mathbb{R}_{<0}} \supseteq \mathbb{C}^\times$$




$Z \times \mathbb{Z}/\sim$ : • $(x, k) \sim (x, k+1)$, $x \in L_+$, $x \in L_-$

• no relations among other pts




$$\begin{aligned}
(z, k) &\xrightarrow{\text{"}\log_k\text{"}} \mathbb{C} \\
\alpha \in U &\longmapsto \log_0(\alpha) + 2\pi i k \\
r \in L_+ &\longmapsto \ln(|r|) + \pi i + 2\pi i k \\
r \in L_- &\longmapsto \ln(|r|) - \pi i + 2\pi i k
\end{aligned}$$

yields $\underset{r \in L_+}{\log_k(r, k)} = \underset{r \in L_-}{\log_{k+1}(r, k+1)}$

$\Rightarrow \log$ is defined on $X = Z \times \mathbb{Z}/\sim$ as a single valued function $\log : X \longrightarrow \mathbb{C}$

The construction of $X$ ensures that we cannot build a loop in $X$ projecting to a loop in $\mathbb{C}^\times$ around 0. This "resolves" the multivalue issue.

We'll see later that $(X, p, \log, \overline{(1,0)})$ is the maximal analytic cnt of $\varphi$ at 1.

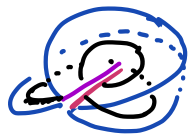
$$\begin{tikzcd}
Z \times \mathbb{Z} \arrow[rr, "pr_1"] \arrow[dr, "\pi"'] & & \mathbb{C}^\times \\
& X \arrow[ur, "p"'] &
\end{tikzcd}$$

② $\sqrt{z}$ We have $Y=\mathbb{C}^\times$, $a=1$ $\varphi$ = series expansion of $z^{1/2}$ near 1

$$\varphi(z) = z^{1/2} = (1-(1-z))^{1/2} = \sum_{n=0}^{\infty} \frac{(1-z)^n}{n!}\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)$$

We know $z^{1/2} = e^{\frac{1}{2}\log z}$ so we can use AC of $\log z$ around 1 to determine that of $z^{1/2}$.

Need to build $F$ that makes these 2 diagram commute



$$\begin{tikzcd} X := \text{Spiral} \arrow[rr, "F"] \arrow[dr, "\pi_1"'] & \arrow[d, phantom, "\circlearrowleft"] & X' \arrow[dl, "q\ (\text{TBD})"] \\ & \mathbb{C}^\times & \end{tikzcd} \quad \& \quad \begin{tikzcd} X := \text{Spiral} \arrow[rr, "F"] \arrow[dr, "\log"'] & \arrow[d, phantom, "\circlearrowleft"] & X' \arrow[dl, "\sqrt{\ } = e^{\frac{1}{2}\log}"] \\ & \mathbb{C}^\times & \end{tikzcd}$$

The space $X'$ of $MAC(z^{1/2}, 1)$ is built by further identifying points in odd, resp. even floors (ie, by the parity of the second coordinate) This gives the map $F$., $q(\bar{x}) = \pi_1(x)$ for $x \in X$.

Q: Alternative construction of $X'$ that shows $X'$ is a RS?

A As a space $X' = \{(t, t^2) : t \in \mathbb{C}^\times\} = V(y^2 - x) \subseteq \mathbb{C}^\times \times \mathbb{C}^\times$

- $q : X' \to \mathbb{C}^\times$ agrees with the second projection $pr_2 : \mathbb{C}^\times \times \mathbb{C}^\times \to \mathbb{C}^\times$
- $g : X' \to \mathbb{C}^\times$ ——— first ——— $pr_1 : \mathbb{C}^\times \times \mathbb{C}^\times \to \mathbb{C}^\times$
- $b = (1,1)$ maps to $1 = a$ under $q$.

**Corollary**: If $(X, p, f, b)$ is an analytic cont of $\varphi \in \mathcal{O}_a$, then $X$ is not compact.

Pf/ $f : X \to \mathbb{C}$ is non-constant & holomorphic. But we cannot have this if $X$ is compact by the Maximum Modulus Principle. □

**Remark**: If we compactify, we'll obtain quadruples where $p$ is a branched covering & $f$ is a meromorphic function. The map $f$ will be proper

## §10.2. Analytic continuations and coverings:

Given $Y$ RS, $a \in Y$, $\varphi \in \mathcal{O}_a$ we build a maximal analytic cont $(X, p, f, b)$ using $X$ = conn comp of $|\mathcal{O}|$ containing $\varphi$, $p : |\mathcal{O}| \to Y$, $b = \varphi$ & $f(\eta) = \eta(p(\eta))$
$\quad\quad \mathfrak{z} \in \mathcal{O}_y \longmapsto y$ $\quad\quad \forall \eta \in X$.

Here is the main result:

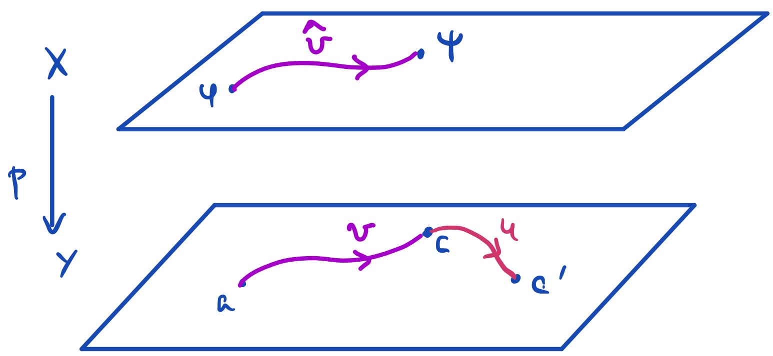
**Theorem 1:** Assume that $\varphi \in \mathcal{O}_a$ admits an analytic continuation along every curve in $Y$ which starts at $a$. Let $(X, p, F, b)$ be the maximal analytic cont. of $\varphi$. Then, $p : X \longrightarrow Y$ is a covering map.

Pf/ We know $p$ is a local homeomorphism & $Y$ is a connected manifold, so $p$ is surjective.

To show $p$ is a covering it is enough to check has the curve lifting property (Thm 1 §5.2), because $X$ is Hausdorff.

Given
$$\begin{tikzcd} & X \arrow[d, "p"] \\ {[0,1]} \arrow[r, "u"] & Y \end{tikzcd}$$
with $u(0) = c \in Y$ we know we have a path $w : [0,1] \to Y$ joining $a$ to $c$. Fix $c' = u(1)$

Pick $\psi \in X$ with $p(x) = c$ This means that $x = \psi = AC_v(\varphi)(1)$ by construction for some path $v$ joining $a$ to $c$ (take $\omega : [0,1] \to X$, $\omega(0) = \varphi$ & $\omega(1) = \psi$. So $v = p \circ \omega$ works!)




Then, we consider the path $w = v * u$. By hyp, this path yields a germ $\eta \in \mathcal{O}_{c'}$, via $\eta = AC_{v*u}\, \varphi$. But $\widehat{v*u}_{(1)} = AC_{v*u}(\varphi)_{(1)} = AC_u\left(AC_v(\varphi)(1)\right)_{(1)}$ since $[0,1]$ is connected. This means: $\hat{u}$ exists & $\hat{u}_{(t)} = AC_u(\psi)_{(t)}$.

**Theorem 2:** Assume that $\varphi \in \mathcal{O}_a$ admits an analytic continuation along every curve in $Y$ which starts at $a$. Let $(X, p, F, b)$ be the maximal analytic cont. of $\varphi$. Then,

(1) $\exists ! F : \tilde{Y} \longrightarrow X$ local homeo with $F(\tilde{a}) = \varphi$ &

$\tilde{a} = (a, [1_a])$

$$\begin{tikzcd} \tilde{a} \in \tilde{Y} \arrow[rr, "F"] \arrow[dr, "\tilde{p}"', "\text{univ covering}"'] & \arrow[d, phantom, "\circlearrowleft"] & X \ni b = \varphi \arrow[dl, "p", "\text{covering}"'] \\ & a \in Y & \end{tikzcd}$$

(2) $\pi_1(X,b) \simeq \mathrm{Deck}(\tilde{Y} \xrightarrow{F} X) < \mathrm{Deck}(\tilde{Y} \mid Y) \simeq \pi_1(Y,a)$

(3) $N(\varphi) = \{\gamma \in \pi_1(Y,a) \mid AC_\gamma(\varphi) = \varphi\} < \pi_1(Y,a)$ gives rise to

$$\begin{tikzcd} \tilde{Y} \arrow[rr, "\pi"] \arrow[dr, "\tilde{p}"'] & \circlearrowleft & X' = \tilde{Y}/_{N(\varphi)} \arrow[dl, "q \text{ covering}"] \\ & Y & \end{tikzcd}$$

with $q(\bar{z}) = \tilde{p}(z)$. .

Pf/ (1) follows from Theorem 1 since $p$ is a covering & $\tilde{p} : \tilde{Y} \longrightarrow Y$ is the universal covering

(2) follows by Theorem 1 in §7.1 & §7.2.

(3) By construction $AC_{\gamma_1 * \gamma_2^-}(\varphi)_{(1)} = AC_{\gamma_2^-}(AC_{\gamma_1}(\varphi)_{(1)})_{(1)}$ $\forall\, \gamma_1, \gamma_2$ curves that can be concatenated. So if $\gamma_1, \gamma_2 \in N(\varphi)$, we get $\gamma_1 * \gamma_2^- \in N(\varphi)$. This says $N(\varphi)$ is a subgroup.

The Galois correspondence gives $X'$, $\pi$ & $q$. □

Natural question: Does $X' = \tilde{Y}/N(\varphi)$ correspond to an analytic continuation of a germ at a pt $y \in Y$? If so, $X'$ would come with a natural non-constant holomorphic function $g : X' \longrightarrow \mathbb{C}$. This need not exist (eg, if $X'$ is compact)

We'll return to this in the future (construction of RS via differential 1-form)

Back to our examples from §10.1:

1. We know that log admits analytic continuations along any path in $\mathbb{C}^\times$ starting at 1. Furthermore, the Spiral space $X = Z \times \mathbb{Z}/_\sim$ is simply connected & $p$ is a covering map but Thm 1, so $X = \tilde{Y} \xrightarrow{\log} Y$ is the universal covering of $Y = \mathbb{C}^*$

2. Similarly, $z^{1/2}$ admits analytic cont along any path in $\mathbb{C}^*$ starting at 1. So, we have

$$\begin{tikzcd} \log_0 z \in X \arrow[rr, "F"] \arrow[dr, "\log"', "\text{univ covering}"] & \circlearrowleft & X' = AC(z^{1/2}) \ni \sqrt{z} \arrow[dl, "\text{``}\sqrt{\ }\text{'' covering}"] \\ & \mathbb{C}^* \ni 1 & \end{tikzcd}$$

with $F(\log_0 z) = \sqrt{z}$

By construction, $F(\overline{(x,k)}) = \overline{(x, k \bmod 2)}$ in $Z \times \{0,1\}/_\sim$

$N(\varphi) = 2\mathbb{Z}$ (parity of winning number of $\varphi$ around 0)

$\mathbb{Z} = \pi_1(\mathbb{C}^*, 1) \simeq \mathrm{Deck}(X|Y)$

$\Rightarrow \mathrm{Deck}(X \xrightarrow{F} X') = \mathbb{Z}_2 \simeq \mathrm{Deck}(X|Y)/_{N(\varphi)}$

$\simeq \pi_1(X', \sqrt{z})$ by strong Galois correspondence (HW3).

**Next goal:** • Build Riemann Surfaces for algebraic germs ( = germs satisfying a polynomial equation) Eg $\varphi = z^{1/2}$ solves $T^2 - z = 0$ in $\mathbb{C}[z][T]$.

• Show a Galois correspondence: $\mathcal{M}(Y) \xrightarrow{p^*} \mathcal{M}(X) \Longleftrightarrow X \xrightarrow{p} Y$ branched holo covering

## §10.3 Elementary symmetric functions

**Setting:** $X \xrightarrow{p} Y$ n-sheeted, unbranched, holomorphic covering map between RS.

$f \in \mathcal{M}(X)$ meromorphic function.

• Given $y \in Y$, pick $y \in V \subseteq Y$ open & $\{U_1, \dots, U_n\}$ opens with

(1) $p^{-1}(V) = \bigsqcup_{i=1}^{n} U_i$

(2) $p|_{U_i} : U_i \xrightarrow{\sim} V$ homeo  Write $q_i = (p|_{U_i})^{-1} : V \longrightarrow U_i$

Then $f_i = f \circ q_i : V \longrightarrow \mathbb{C}$ is meromorphic

• Fix $T$ indeterminate, & write $\prod_{i=1}^{n}(T - f_i) = T^n + c_1 T^{n-1} + \dots + c_n$

where $c_i \in \mathcal{M}(V)$

**Lemma 1:** $c_i = (-1)^i s_i(f_1, \dots, f_n)$ where $s_i$ = $i^{th}$ elem symm func on n vars

$\left(s_i(\underline{x}) := \sum_{1 \le j_1 < \dots < j_i \le n} x_{j_1} \cdots x_{j_i}\right)$

**Lemma 2:** The construction is independent on the choice of $y \in Y$ in the nbhd $V$ (ie functions will agree on the overlaps & labelling of $U_i$'s is irrelevant)

$\Rightarrow$ They glue to functions $c_1, \dots, c_n \in \mathcal{M}(Y)$

**Def:** We call $c_1, \dots, c_n$ the "elementary symmetric functions of $f$ wrt the covering $p$".

**Q**: What happens if $p$ is branched?

branched ($\Rightarrow$ proper)

**A**: Everything works out in the same way!

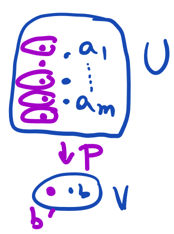
**Theorem 1**: Fix $X, Y$ RS & $p: X \longrightarrow Y$ a proper non constant holo map of degree $n$. Fix $B \subseteq Y$ closed discrete set containing all critical values of $P$, & write $A := p^{-1}(B)$ (closed & discrete, contains all branch points of $p$)
Fix $f \in \mathcal{O}(X \setminus A)$ & let $c_1, \dots, c_n \in \mathcal{O}(Y \setminus B)$ be the elem symm functions of $f$ wrt $p$. Fix $b \in B$. Then, $f$ can be continued to $X$ holomorphically, resp meromorphically, to $p^{-1}(b)$ if, and only if, all $c_1, \dots, c_n$ can be cont holo, resp mero to $b$.

**Pf/ Idea**: $B$ is closed & discrete, so we can connect points in $Y$ along paths in $Y' \setminus B$. The corresponding $c$'s will match.

- We fix $b \in B$ & $\underset{\text{mult}}{p^{-1}(b)} = \{\underset{\nu_1}{a_1}, \dots, \underset{\nu_m}{a_m}\}$ $m \leq n$ with $\sum_{i=1}^{m} \nu(a_i, p) = n$

Since $Y$ is locally compact & $B$ is discrete, we can find $(V, \varphi)$ coord nbhd of $b$ with $\varphi: V \xrightarrow{\sim} \mathbb{D}$, $b \longmapsto 0$, $\overline{V}$ compact & $V \cap B = \{b\}$. Write $\boxed{V^* = V \setminus \{b\}}$

Then $U = p^{-1}(V)$ is a relatively compact nbhd of each $a_1, \dots, a_m$.
Write $\boxed{U^*} = U \setminus p^{-1}(b) = U \setminus \{a_1, \dots, a_m\}$




- We do the holomorphic & meromorphic cases separately:

① **Holomorphic Case**:

Pick a point in $V \setminus B$, write $c_i$'s using $s_i$ in $n$ variables

($\Rightarrow$) If $f$ can be continued holomorphically to all $a_1, \dots, a_m$ $\Longrightarrow$ The corresponding $f_i$'s can be continued from $V^*$ to $V$ $\Rightarrow$ $c_i$'s can be continued from $V \setminus \{b\}$ to $b$ ($s_i(f_1, \dots, f_n)$ will be bounded on $V \setminus \{b\}$).

Moreover $c_i(b) = (-1)^i s_i(\underbrace{f_1(b), \dots, f_{\nu_1}(b)}_{\nu_1}, \dots\dots, \underbrace{f_{n-\nu_m}(b), \dots, f_n(b)}_{\nu_m})$.

($\Leftarrow$) Conversely, if all $c_i$'s can be continued holomorphically to $b$ then all $c_i$'s are bounded on $V_1 \setminus \{b\}$. Then: all $c_1(p(x)), \dots, c_n(p(x))$ are bounded in $U^*$

Using the identity: $\boxed{f^n_{(x)} + c_1(p(x)) f^{n-1}_{(x)} + \dots + c_n(p(x)) = 0}$ on $U^*$, we get that (*)

$f$ is also bounded on $U^*$. By Removable Sing Thm, $f$ can be extended holomorphically to $\{a_1, \dots, a_n\} = p^{-1}(b)$.

② <u>Meromorphic Case</u>:

We proceed in a similar fashion, but now we need to multiply $f$, resp $c_1, \dots, c_n$ by appropriate monomials & check boundedness on $U^*$, resp $V^*$.

- For $V^*$ use the local coordinate $z$ on $\mathbb{D} \simeq V$
- For each comp of $U^*$ use $\varphi = p^*_{|U_i}(z) \in \mathcal{O}(U_i)$ vanishes at $a_j \in U_i$.

Write $\varphi = p^*(z)$, so $\varphi$ vanishes on $p^{-1}(b)$.

($\Rightarrow$) Assume $\varphi^k f$ can be continued holomorphically to $a_1, \dots, a_m$

Then, our holomorphic calculation says $z^{ki} c_i$ can be continued holomorphically to $b$. So $c_i$ can be continued meromorphically to $b$

($\Leftarrow$) For each $i$ we have $k_i \geq 0$ such that $z^{k_i} c_i$ can be cont'd holomorphically to $b$. Take $k := \max\{k_1, \dots, k_m\} \Rightarrow z^{ki} c_i \in \mathcal{O}(V)$

Multiplying the identity (*) by $\varphi^{kn}$ we see that:

$$0 = \varphi^{kn} (*) = (\varphi^k f)^n + (\varphi^k (c_1 \circ p))(\varphi^k f)^{n-1} + \dots + \varphi^{kn}(c_n \circ p) \text{ on } \mathcal{O}(U^*)$$

Here $\varphi^{ki}(c_i \circ p)_{(x)} = p^*(z^{ki} c_i)$, so it is bounded in $\mathcal{O}(U^*)$

By the holomorphic case, $\varphi^k f$ can be extended holomorphically to $U$.

So $f$ can be extended meromorphically to $U$. □

<u>Remark</u>: We don't need the condition that $X$ is connected. So we can take $X = \bigsqcup_i X_i$ $X_i$ RS