Lecture XI: Algebraic functions

Last Time: ① $p: X \longrightarrow Y$ unbranched proper holo map of degree n

$f \in \mathcal{M}(X) \rightsquigarrow c_1, \ldots, c_n \in \mathcal{M}(Y)$ elem symm functions of f

On V open Y covered by $\bigsqcup_{i=1}^{n} U_i$, set $f_i = f \circ q_i$ where $q_i := (p_{|U_i})^{-1}$ $\rightsquigarrow$ $c_i = (-1)^i s_i(f_1, \ldots, f_n)$

$s_i :=$ elem symm fnc on n letters.

(f satisfies $f^n + p^*c_1 f^{n-1} + \cdots + p^*c_n = 0$.)

② If $p: X \longrightarrow Y$ is a proper holo map of degree n & $B \subseteq Y$ is closed discrete, containing the crit values of p, set $A := p^{-1}(B)$. Pick $f \in \mathcal{O}(X \setminus A)$ & write $c_1, \ldots, c_n \in \mathcal{O}(Y \setminus B)$ for the elem symm functions of f rel to $p|_{X \setminus A}: X \setminus A \longrightarrow Y \setminus B$. Then:

f extends holomorphically to $p^{-1}(b)$ $\Longleftrightarrow$ $c_1, \ldots, c_n$ extend holomorphically to b
(meromorphically) (meromorphically)

Key: f satisfies $f^n + p^*c_1 f^{n-1} + \cdots + p^*c_n = 0 \in \mathcal{M}(U^*)$ for $V \xrightarrow{\sim} \mathbb{D}$, $b \longmapsto 0$ $\quad U = p^{-1}(V)$, $U^* = p^{-1}(V^*)$

TODAY: Build a RS X, a proper holo map $p: X \longrightarrow Y$ & $f \in \mathcal{M}(X)$ algebraic over $p^*\mathcal{M}(Y) \subseteq \mathcal{M}(X)$

§11.1 Branched covering and field extensions:

Recall: $X \xrightarrow{p} Y$ proper non-const holomorphic map of degree n.

$\rightsquigarrow p^*_U : \mathcal{M}(U) \longrightarrow \mathcal{M}(p^{-1}(U))$ for $U \subseteq Y$ open (s∘p is meromorphic on $p^{-1}(U)$ by construction)

$s \longmapsto s \circ p$

We get $p^*: \mathcal{M}(Y) \longrightarrow \mathcal{M}(X)$ map of rings & both $\mathcal{M}(X)$ & $\mathcal{M}(Y)$ are fields (X, Y are connected)

Theorem: Given $X \xrightarrow{p} Y$ proper non-const holomorphic map of degree n, $f \in \mathcal{M}(X)$ & $c_1, \ldots, c_n \in \mathcal{M}(Y)$ elementary symmetric functions of f rel to p, we have:

$$f^n + (p^*c_1) f^{n-1} + \cdots + (p^*c_n) = 0 \qquad (**)$$

The monomorphism $p^*: \mathcal{M}(Y) \longrightarrow \mathcal{M}(X)$ is an algebraic field extension of deg $\leq n$

Moreover, if $\exists f \in \mathcal{M}(X)$ & $y \in Y$ (non crit value) with $p^{-1}(y) = \{x_1, \ldots, x_n\}$

st $f(x_i) \neq f(x_j)$ $\forall i \neq j$, then $[\mathcal{M}(X) : \mathcal{M}(Y)] = n$.

Remark: We'll see later that the pair (f, y) always exists.

Proof. The identity $(**)$ follows from the definition of $c_1, \ldots, c_n \in \mathcal{M}(X \setminus p^{-1}(\text{crit values}))$

- Write $L := \mathcal{M}(X)$, $K = p^* \mathcal{M}(Y) \subset L$. By construction, we see that $\min(f, K)$ has degree $\leq n$ for any $f \in L$, so $L|K$ is algebraic.

- Pick n_0 & $f_0 \in L$ where $n_0 = [K(f_0) : K]$ maximal.

Pick any $f \in L$ & fix $L' = K(f_0, f) \subseteq L$. Then $[L' : K] < \infty$ & by the Primitive Element Theorem $(\mathbb{C} \subseteq K)$ we can find $g \in L' \subseteq L$ with $L' = K(g)$

Then $[K(g) : K] = [K(g) : K(f_0)]\,[K(f_0) : K]$, so by maximality of n_0 $[K(g) : K(f_0)] = 1$ ie $f \in K(f_0)$.

Conclude: $L = K(f_0)$ & $[L : K] = n_0 \leq n$

- Finally, if $n_0 < n$, then $(*)$ $\min(f, K) = f^{n_0} + p^*(\alpha_1) f^{n_0 - 1} + \cdots + p^*(\alpha_{n_0}) = 0$ & when evaluated at $x \in p^{-1}(y)$ $f(x)$ can have at most n_0 values (soln to $(*)$ eval at y)

This contradicts the hypotheses on the pair (f, y).

§11.2 Algebraic Functions

Next goal: Build Riemann surfaces & branched coverings of Y from $Q \in \mathcal{M}(Y)[T]$ (monic)

Theorem 1: Fix Y a RS & $Q = T^n + c_1 T^{n-1} + \cdots + c_n \in \mathcal{M}(Y)[T]$. Assume that Q is irreducible. Then $\exists$ a triple (X, p, f) where:

(1) X is a RS

(2) $p: X \longrightarrow Y$ is a proper non-const holomorphic map of degree n ("n-sheeted branched covering")

(3) $f \in \mathcal{M}(X)$ with $\underbrace{(p^*Q)}_{\in p^*\mathcal{M}(Y)[T] \subseteq \mathcal{M}(X)[T]}(f) = 0$.

satisfying the following universal property: If (Z, q, g) is another such triple, then $\exists!$ $\sigma: Z \longrightarrow X$ biholomorphic with

$$\begin{tikzcd} Z \arrow[rr, "\sigma"] \arrow[dr, "q"'] & \arrow[d, phantom, "\circlearrowleft"] & X \arrow[dl, "p"] \\ & Y & \end{tikzcd} \quad \& \quad \begin{tikzcd} Z \arrow[rr, "\sigma"] \arrow[dr, "g"'] & \arrow[d, phantom, "\circlearrowleft"] & X \arrow[dl, "f"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

(ie fiber preserving + $\sigma^* f = g$)

Definition: (X, p, f) is called the "algebraic function" defined by Q.

Main example: $Y = \mathbb{P}^1$, so $\mathcal{M}(Y) = \mathbb{C}(y)$ $\Rightarrow$ $p: X \longrightarrow \mathbb{P}^1$ is proper (branched holomorphic covering with finite fibers), so X is also compact.

Idea: Build $X' \xrightarrow{p} Y'$ holomorphic covering & $f \in \mathcal{M}(X')$, for $Y' = Y \setminus B$ B closed & discrete & then "fill the holes" in X' & Y' to get $X \xrightarrow{p} Y$ deg n proper holomorphic map & $f \in \mathcal{M}(X)$.

We'll use our classification of unbranched proper coverings of $\mathbb{D}^*$ (see §8.1)

Next, we write the key lemma that allows us to restrict to unbranched proper holo maps. (ie "how to fill holes")

Key Lemma: Fix Y RS, $B \subseteq Y$ closed & discrete set & set $Y' = Y \setminus B$. Assume we are given $p': X' \longrightarrow Y'$ proper unbranched holomorphic covering. Then, p' extends to a branched proper holomorphic covering of Y, ie we have

(1) a RS X

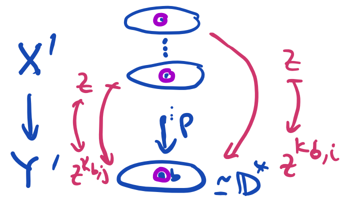
(2) a proper holomorphic map $p: X \longrightarrow Y$

(3) a biholomorphism $X \setminus p^{-1}(B) \xrightarrow{F} X'$ with $p' \circ F = p|_{X \setminus p^{-1}(B)}$

Proof: $\underset{\text{(around } b \in B)}{\text{Locally}}$ $Y' \simeq \mathbb{D}^*$, $p'^{-1}(\mathbb{D}^*) \xrightarrow{p'} \mathbb{D}^*$ proper covering ($\Rightarrow k_b$ sheeted) & we want to extend to $p: X \longrightarrow \mathbb{D}^*$ for some $X = X' \cup \{\text{finitely many pts}\}$

Issue: $p'^{-1}(\mathbb{D}^*)$ will be disconnected, but it will have finitely many connected components. Adding one pt to each component will make each homeomorphic to $\mathbb{D}$.

• These "filled" connected components will have charts compatible with those in X' (because we've classified proper holo maps to $\mathbb{D}$, unbranched over $\mathbb{D}^*$ in Thm 2 §8.3) & the filling pt will map to b under p



• Formally, since B is closed & discrete we can pick coordinate charts $\{(U_b, \varphi_b)\}$ around $b \in B$ with $\varphi_b : U_b \longrightarrow \mathbb{D}$, $b \longmapsto 0$ such that

(1) $U_b \cap B = \{b\} \quad \forall b \in B$

(2) $U_b \cap U_{b'} = \emptyset \quad \forall b \neq b'$

Consider the punctured nbhds $\boxed{U_b^* := U_b \setminus \{b\}}$ for $b \in B$ $(U_b^* \simeq \mathbb{D}^*)$

• Since $p' : Y' \longrightarrow X'$ is proper, $(p')^{-1}(U_b^*)$ has finitely many connected components say n_b many (all open). Write $(p')^{-1}(U_b^*) = \bigsqcup_{j=1}^{n_b} V^*_{b,j}$

Restrict p' to $V^*_{b,j}$ = $p'_{b,j} := p'|_{V^*_{b,j}} : V^*_{b,j} \longrightarrow U_b^* \simeq \mathbb{D}^*$ is unbranched proper holo non-const map on R.S, so it is a covering. Let $k_{b,j}$ be its number of sheets

By Theorem 1 §8.3 we find biholomorphic maps $\psi_{b,j} : V^*_{b,j} \longrightarrow \mathbb{D}^*$ st

$$\begin{tikzcd}
V^*_{b,j} \arrow[r, "\psi_{b,j}"] \arrow[d, "p'_{b,j}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{D}^* \arrow[d] & z \arrow[d, mapsto] \\
U_b^* \arrow[r, "\varphi"] \arrow[d, hook] & \mathbb{D}^* \arrow[d, hook] & z^{k_{b,j}} \\
U_b \arrow[r, "\varphi"] & \mathbb{D} &
\end{tikzcd}$$

(***)

We can easily "fill in the holes" by adding 1 pt to each $V^*_{b,j}$, which will map to 0 under the extension of $\psi_{b,j}$ to $\boxed{V_{b,j} := V^*_{b,j} \sqcup \{y_{b,j}\}}$

We have a unique Top on $V_{b,j}$ that makes $\psi_{b,j} : V_{b,j} \longrightarrow \mathbb{D}$, $y_{b,j} \longmapsto 0$, into a homeo map. This is a biholomorphism & it is compatible with the charts on X'.

The compatibility with the charts $(V_{b,k}, \psi_{b,k})$ & $(V_{b',i}; \psi_{b',i})$ is tautological.

- We define $\boxed{X = X' \sqcup \bigsqcup_{b\in B} \{y_{b,j} : j=1,\dots,n_b\}}$

- Topology on X has basis $\mathcal{B}(X') \cup \bigcup_{b\in B} \bigcup_{j=1}^{n_b} \{ \underbrace{(\{y_{b,j}\} \cup ((p')^{-1}(W) \cap V^{*}_{b,j}))}_{\text{nbhds of } y_{b,j}} : W \subseteq X \text{ open} \}$ with $b\in W$

With this topology, X is Hausdorff. (Y is Hausdorff & X' is Hausdorff)

- X is connected & charts described above give a complex structure to X. ($\overline{X'} = X$ & X' is connected)

- Define $p : X \longrightarrow Y$ via $p|_{X'} = p'$ & $p(y_{b,j}) = b$ $\forall b\in B$ $j=1,\dots,n_b$

<u>Claim</u>: p is continuous, holomorphic & proper of degree n ($=\deg p'$)

Pf/. p' is cont, holo & $p|_{V_{b,j}} : V_{b,j} \longrightarrow U_b$ is continuous & holomorphic by (***)

$\Rightarrow p$ is cont & holo.

To show: $p^{-1}(K)$ is compact $\forall K \subseteq Y$ compact we use that $K \cap B$ is finite, and the identity: $p^{-1}(K) = \underbrace{(p')^{-1}(K')}_{\text{compact}} \cup \underbrace{p^{-1}(K \cap B)}_{\text{finite}}$ for $K' = K \cap X'$ ($K' \subseteq X'$ is compact)

- $X' = X \setminus p^{-1}(B)$, so we pick $F = id_{X'}$

§11.3 Proof of Existence in Theorem 1 §11.2

We proceed in several steps

<u>STEP 1</u>: We use the <u>discriminant</u> Δ of Q (in the variable T) to find a closed discrete set $B \subseteq Y$ where $c_i \in \mathcal{O}(Y \setminus B)$ $\forall i$ & for each $y \notin B$,

$$Q_y(T) := T^n + c_1(y)\, T^{n-1} + \dots + c_n(y) \in \mathbb{C}[T]$$

has n distinct roots in $\mathbb{C}$

<u>Definition</u>: $\Delta \in \mathcal{M}(Y)$ is a polynomial in $c_1, \dots, c_n$ that vanishes

$\Leftrightarrow Q$ is reducible in $\mathcal{M}(Y)[T]$ (equiv, if Q has a double root in $\overline{\mathcal{M}(Y)}$)

Obs: Δ can be computed as $\Delta = (-1)^{\frac{n(n-1)}{2}} \operatorname{Res}_T(Q, Q') = (-1)^{\frac{n(n-1)}{2}} \prod_{1\le i<j\le n} (r_i - r_j)^2$ if $Q = \prod_{i=1}^{n} (T - r_i)$ is the factorization over $\overline{\mathcal{M}(Y)}$.

- Consider $\boxed{B} := \{ y \in Y : \Delta_{(y)} = 0 \} \cup \bigcup_{i=1}^{n} \{ y \in Y : y \text{ is a pole of } c_i \}$

By construction, B is closed & discrete & for $Y' = Y \setminus B$ we have

(1) $c_i|_{Y'} \in \mathcal{O}(Y')$ $\forall i = 1, \ldots, n$

(2) $\Delta_{(y)} \neq 0$ $\forall y \in Y'$ $\quad (\Delta \in \mathcal{O}(Y')$ since $\Delta \in \mathbb{Z}[c_1, \ldots, c_n].)$

Claim 1: $Q_y(T) \in \mathbb{C}[T]$ has n distinct roots in $\mathbb{C}$ (from the def of Δ & Y')

STEP 2: We build X' a RS & $p' : X' \longrightarrow Y'$ n-sheeted (unbranched) holo covering

We construct X' using $\begin{array}{c} |\mathcal{O}| \\ \downarrow p \\ Y \end{array}$ & the n roots of each $Q_y(T)$.

Claim 2: $X' := \{ \eta = (y, \eta) : y \in Y', \eta \in \mathcal{O}_y \text{ \& } Q_y(\eta) = 0 \}$

[Here we view $Q_y(T) \in \mathcal{O}_y[T]$ & we need to find solutions to $Q_y(T) = 0$. We'll see this can be done in Lemma 1 & Corollary 1 below)

- Set $p' : X' \longrightarrow Y'$
  $\eta \longmapsto y$ if $\eta \in \mathcal{O}_y$

Claim 3: p' is an n-sheeted unbranched covering (see Lemma 2 below)

STEP 3: Assume X' is connected & build X RS & $p : X \longrightarrow Y$ extending $p' : X' \longrightarrow Y'$ using Key Lemma §10.5. p is n-sheeted branched proper holo map.

STEP 4: Build $f \in \mathcal{M}(X)$ with $(p^*(Q))(f) = 0$
$\in \mathcal{M}(X)[T]$

First define $f : X' \longrightarrow \mathbb{C}$ with $f(\underset{\in \mathcal{O}_y}{\eta}) = \eta(p'(\eta)) = \eta(y)$

Claim 4: f is holomorphic

Pf/ Given $\eta \in X'$ (connected), find $U \subseteq Y'$ open & $s \in \mathcal{O}(U)$ with $\eta \in N(U,s) \subseteq X'$
$(\eta \in \mathcal{O}_y)$ $\quad y \in U$

Then $f|_{N(U,s)} = s \circ p'|_{N(U,s)}$ is holomorphic □

• Take $(p')^* Q_{(T)} = T^n + ((p')^* c_1)\, T^{n-1} + \cdots + (p')^* c_n$

$$= T^n + c_1 \circ p'(x)\, T^{n-1} + \cdots + c_n \circ p'(x) \in \mathcal{M}(X')[T]$$

<u>Claim 5</u>: $(p')^* Q(f) = 0$ (clear from the def of X' and f)

<u>Claim 6</u>: f extends uniquely to $f \in \mathcal{M}(X)$ & $p^* Q(f) = 0$

Pf/ We know $\{c_i \circ p'\}$ extend meromorphically to $\{c_i \circ p\}$, so by Theorem 1 §10.3 f extends uniquely to a meromorphic function on X & $(p^* Q)(f) = 0$

<u>STEP 5</u>: Show that X' is connected

Assume the contrary and work with the finitely many connected comp of X', say $X'_1, \ldots, X'_s$ with $s \leq n$ (because p' is n-sheeted).

We can repeat STEPS 3 & 4 for each X'_i $i = 1, \ldots, s$ & find

$\begin{cases} \bullet\ X_i \xrightarrow{p} Y \text{ branched covering with } n_i\text{-sheets extending } p'|_{X'_i} : X'_i \to Y'_i \ (n_i \text{ sheeted}) \\ \bullet\ f_i \in \mathcal{M}(X_i) \text{ with } (p_i^* Q)(f_i) = 0 \end{cases}$

Here, $\sum_{i=1}^{s} n_i = n$ by construction.

Using the elementary symmetric functions of f_i we can build $Q_i \in \mathcal{M}(Y)[T]$ of degree n_i satisfying

$$Q(T) = Q_1(T)\, Q_2(T) \cdots Q_s(T)$$

(because (RHS) $\mid Q(T)$ & is monic of degree n)

This contradicts the assumption that Q is irreducible unless $s = 1$ ($\Rightarrow X'$ is conn)

□

To complete the proof of Theorem 1 we need some technical results to settle Claims 2 and 3. Lemma 1 & Corollary 1 below are used for Claim 2, whereas Lemma 2 confirm Claim 3.

The first lemma says we can find a germ solving a polynomial in $\mathcal{O}_y[T]$ if we can solve the polynomial in $\mathbb{C}[T]$ obtained by evaluating at y using a simple root.

Lemma 1: Fix $c_1, \dots, c_n \in \mathcal{O}(D_R(0))$ with $D_R(0) = \{z : |z| < R\}$ & assume

$$Q_0(T) = T^n + c_1(0)\, T^{n-1} + \dots + c_n(0) \in \mathbb{C}[T]$$

has a simple root, say w_0. Then, $\exists\, r$ with $0 < r < R$ & $! \varphi \in \mathcal{O}(D_r(0))$ with $\varphi(0) = w_0$ & $Q(\varphi) := \varphi^n + c_1 \varphi^{n-1} + \dots + c_n = 0$ in $\mathcal{O}(D_r(0))[T]$

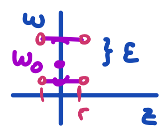
($\Rightarrow$ $\rho_0(\varphi)$ solves $\rho_0(Q)[T] = 0$ in $\mathcal{O}_{D_R(0),0}$)

Proof. We consider $w \in \mathbb{C}$ & $z \in D_R(0)$ & the polynomial

$$F(z,w) := w^n + c_1(z)\, w^{n-1} + \dots + c_n(z).$$

Since $F(0,w_0) = 0$ & w_0 is a simple root of $F(0,w)$ we can find $\varepsilon > 0$ such that $F(0,\underline{w})$ has only 1 zero in $\overline{D_\varepsilon(w_0)} = \{w : |w - w_0| \leq \varepsilon_0\}$

(the zeroes of $F(0,\underline{w})$ are discrete)



By continuity of F & compactness of $\{|w-w_0| = \varepsilon\}$, $\exists\ 0 < r < R$ st $F_{(z,w)}$ has no zeroes on:

$$\{(z,w) : \quad z \in D_r(0) \quad \& \quad |w - w_0| = \varepsilon\}$$

- We will build φ by Residue type Formula, using the circumference $C = \{|w - w_0| = \varepsilon\}$

First, we define $\eta : D_r(0) \longrightarrow \mathbb{C}$ as

$$\eta(z) := \frac{1}{2\pi i} \oint_C \frac{\partial_w F(z,w)}{F(z,w)}\, dw = \#\text{ of soln of } F(z,T) \text{ in } |T - w_0| < \varepsilon$$

[**Reason**: if $P(t) = \prod_{j=1}^{\ell} (t - \alpha_j)^{n_j} \Rightarrow \dfrac{P'(t)}{P(t)} = \sum_{j=1}^{\ell} \dfrac{n_j}{t - \alpha_j}$ & $\dfrac{1}{2\pi i} \oint_C \dfrac{P'(t)}{P(t)}\, dt = \sum_{\substack{j=1 \\ \alpha_j \in \mathrm{Int}(C)}}^{\ell} n_j$]

By construction η is continuous and $\mathrm{im}(\eta) \subseteq \mathbb{Z}$, so it's constant

Since $\eta(0) = \mathrm{mult}(w_0, F_{(0,w)}) = 1$ by our choice of ε, we get $\boxed{\eta(z) \equiv 1.}$

We define $\varphi: D(0,r) \longrightarrow \mathbb{C}$ via $\varphi(z) := \frac{1}{2\pi i} \oint_C w \frac{\partial_w F(z,w)}{F(z,w)} dw$

Claim. $\varphi(z)$ = the value w of the unique zero of $F(z,w)$ for $w \in D_\varepsilon(w_0)$.

Pf/ If $P(t) = \prod_{j=1}^{\ell} (t-\alpha_j)^{n_j} \Rightarrow \frac{1}{2\pi i} \oint_C \frac{t P'(t)}{P(t)} dt = \sum_{\substack{j=1 \\ \alpha_j \in \mathrm{Int}(C)}}^{\ell} n_j \alpha_j$

But since $\eta(z) = 1$, we know $F(z,w)$ only has 1 zero in $D_\varepsilon(w_0)$, so $\varphi(z)$ collects its value.

- $\varphi(0) = w_0$ by construction
- φ depends holomorphically on z so $\varphi \in \mathcal{O}(D_r(0))$.

The construction gives uniqueness for free! (The integral formula computes the solution by the Residue Theorem). $\square$

Corollary 1: Consider $Q(T) := T^n + c_1 T^{n-1} + \dots + c_n \in \mathcal{O}_y[T]$ & assume $q(T) = T^n + c_1(y) T^{n-1} + \dots + c_n(y) \in \mathbb{C}[T]$ has n distinct roots $w_1, \dots, w_n$. Then, there exist $[\varphi_1], \dots, [\varphi_n] \in \mathcal{O}_y$ with

$$\varphi_k(y) = w_k \quad \& \quad Q([\varphi_k]) = 0 \quad \forall k = 1, \dots, n$$

Proof: Lift $c_1, \dots, c_n \in \mathcal{O}_y$ to an open $U_0 \ni y$. Consider a coord chart (U, ψ) of y with $U \subseteq U_0$ & $\psi: U \xrightarrow{\sim} D$, $y \longmapsto 0$.

Use Lemma 1 to build φ_j for each w_j. Take $r = \min\{r_1, \dots, r_n\} < 1$ & identify $D_r(0)$ with an open $U' \ni y$ inside U. Then $Q([\varphi_k]) = 0$ $\square$

Lemma 2: $p': X' \longrightarrow Y'$ is an n-sheeted covering map. (This is Claim 3)

Proof We prove this by the definition of covering. Pick any $y \in Y'$

By construction $(p')^{-1}(y)$ has n elements, build using Corollary 1.

Call them $f_1, \ldots, f_n \in \mathcal{O}_{Y',y}$. So $\rho_y(Q(T)) = \prod_{i=1}^{n} (T - f_i)$ in $\mathcal{O}_y[T]$

Lift $f_1, \ldots, f_n$ to $s_1, \ldots, s_n$ in $\mathcal{O}(V)$ for some open $V \subseteq Y'$, $y \in V$. Then

$Q(T) = \prod_{i=1}^{n} (T - s_i)$ in $\mathcal{O}(V)[T]$

By construction, we have $(p')^{-1}(V) = \bigcup_{i=1}^{n} \mathcal{N}(V, s_i)$

<u>Claim 1</u>: $\mathcal{N}(V, s_i) \cap \mathcal{N}(V, s_j) = \emptyset \quad \forall i \neq j$

Pf/ $f_i \neq f_j$, so $s_i \neq s_j$ in $\mathcal{O}(V)$. Furthermore, by the identity Thm

$\rho_{y'}[s_i] \neq \rho_{y'}[s_j] \quad \forall y' \in$.

<u>Claim 2</u> $p'|_{\mathcal{N}(U, s_i)} : \mathcal{N}(V, s_i) \longrightarrow V$ is a homeo.

(we saw this in the proof of Thm 1 §8.2)

So the open V of Y' & the collection $\{\mathcal{N}(V, s_i)\}_{i=1}^{n}$ satisfy the def of covering for y. □