Lecture XII: Algebraic functions II

Recall $p : X \to Y$ proper holo deg $n \Rightarrow [\mathcal{M}(X) : p^*\mathcal{M}(Y)] \leq n$ & $=$ if $\exists f \in \mathcal{M}(X)$ & $y \in Y$ with n values on $p^{-1}(y)$

Theorem 1: Fix Y a RS & $Q = T^n + c_1 T^{n-1} + \dots + c_n \in \mathcal{M}(Y)[T]$.

Assume that Q is irreducible. Then $\exists$ a triple (X, p, f) where:
("algebraic function")

(1) X is a RS

(2) $p : X \to Y$ is a proper non-const holomorphic map of degree n ("n-sheeted branched covering")

(3) $f \in \mathcal{M}(X)$ with $\underbrace{(p^*Q)}_{\in p^*\mathcal{M}(Y)[T] \subseteq \mathcal{M}(X)[T]}(f) = 0$.

satisfying the following universal property: If (Z, q, g) is another such triple, then $\exists!\ \sigma : Z \to X$ biholomorphic with

$$\begin{tikzcd} Z \arrow[rr, "\sigma"] \arrow[dr, "q"'] & \circlearrowleft & X \arrow[dl, "p"] \\ & Y & \end{tikzcd} \quad \& \quad \begin{tikzcd} Z \arrow[rr, "\sigma"] \arrow[dr, "g"'] & \circlearrowleft & X \arrow[dl, "f"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

(ie fiber preserving + $\sigma^* f = g$)

Last Time: we proved existence by first avoiding discriminant of Q & poles of c_i $\left(\begin{smallmatrix} X' \subseteq X \\ \downarrow p \\ Y' \end{smallmatrix}\right)$

Idea: Pick $B = \{\text{zeros of discriminant of } Q\} \cup \bigcup_{i=1}^{n} \{\text{poles of } c_i\}$ so

on $Y' = Y \setminus B$: (1) $Q(T, y)$ has simple roots in $\mathbb{C}$ for each $y \in Y'$

(2) $c_i|_{Y'} \in \mathcal{O}(Y')$ $\forall i = 1, \dots, n$

- Build $X' \subseteq |\mathcal{O}_{Y'}|$ via $X' = \{(y, \eta) : y \in Y', \eta \in \mathcal{O}_{Y',y} \ \& \ \rho_y Q(\eta) = 0\}$ (✱)
  ($\rho_y Q = T^n + \rho_y(c_1) T^{n-1} + \dots + \rho_y(c_n) \in \mathcal{O}_{Y',y}[T]$.)
- $X' \xrightarrow{p} Y'$ $\quad p(\eta) = y$ $\quad$ n-sheeted covering (Claim)
- $X' \xrightarrow{f} \mathbb{C}$ $\quad f(\eta) = \eta(y) = \eta(p(\eta))$ $\quad (f \in \mathcal{O}(X')) \Rightarrow (p^*Q)(f) = 0$.

Assuming X' is connected (it follows because Q is irred.) we can extend p to

$$\begin{tikzcd} X \arrow[r, "p"] & Y \\ X' \arrow[u, hook] \arrow[r, "p"] & Y' \arrow[u, hook] \end{tikzcd}$$

& show p is holo, proper of degree n (✱)

- f will extend to X meromorphically since $c_1 \dots c_n$ are the elem sym functions of F on Y' & the extend meromorphically to Y. $\Rightarrow f$ extends to $\mathcal{M}(X)$.

For (*): Show $p|_{X'}$ is a covering with n sheets & use Key Lemma §11.2 To get $p: X \to Y$

① Given $y \in Y'$, let $\{w_1, \dots w_n\}$ be the n roots of $\rho_y Q(T,y) \in \mathbb{C}[T]$.

$\rightsquigarrow$ Build $\varphi_1, \dots, \varphi_n \in \mathcal{O}_{Y',y}$ with $\rho_y Q(\varphi_i) = 0 \quad \forall i = 1 \dots n$

$$\varphi_i(y) = w_i$$

$\Rightarrow$ Lift to $s_1, \dots, s_n \in \mathcal{O}_{Y'}(U)$ for $U \ni y$ open in Y' connected, so $Q(s_i, y) = 0 \quad \forall i = 1 \dots n, \ y \in U$

$\Rightarrow p^{-1}(U) = \bigsqcup_{i=1}^{n} \mathcal{N}(U, s_i) \subseteq X'$ & we know $p|_{\mathcal{N}(U,s_i)} : \mathcal{N}(U,s_i) \longrightarrow U$ is homeo.

($\rho_{y'}(s_i) \neq \rho_{y'}(s_j) \quad \forall y' \in U$ by the identity thm. ($\rho_y(s_i) = \varphi_i \neq \varphi_j = \rho_y(s_j)$)).

② The key lemma from Lecture 11 (§11.2 says extension $p: X \to Y$ is proper, holo of degree n).

To finish building X', we need to prove (*) We can do it for $Y' = D_R(0)$ & $y = 0$:

Lemma 1: Fix $c_1, \dots, c_n \in \mathcal{O}(D_R(0))$ with $D_R(0) = \{z : |z| < R\}$ & assume

$$Q_0(T) = T^n + c_1(0)\, T^{n-1} + \dots + c_n(0) \in \mathbb{C}[T]$$

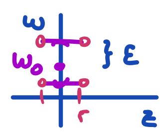
has a simple root, say w_0. Then, $\exists\, r$ with $0 < r < R$ & $!\varphi \in \mathcal{O}(D_r(0))$ with $\varphi(0) = w_0$ & $Q(\varphi) := \varphi^n + c_1 \varphi^{n-1} + \dots + c_n = 0$ in $\mathcal{O}(D_r(0))[T]$
($\Rightarrow \rho_0(\varphi)$ solves $\rho_0(Q)[T] = 0$ in $\mathcal{O}_{D_R(0),0}$)

Proof. We consider $w \in \mathbb{C}$ & $z \in D_R(0)$ & the polynomial

$$F(z,w) := w^n + c_1(z)\, w^{n-1} + \dots + c_n(z). \in \mathbb{C}[z,w]$$

Since $F(0, w_0) = 0$ & w_0 is a simple root of $F(0,w)$ we can find $\varepsilon > 0$ such that $F(0,w)$ has only 1 zero in $\overline{D_\varepsilon(w_0)} = \{w : |w - w_0| \le \varepsilon_0\}$

(the zeroes of $F(0,w)$ are discrete)



By continuity of F & compactness of $\{|w - w_0| = \varepsilon\}$, $\exists\ 0 < r < R$ st $F_{(z,w)}$ has no zeroes on:

$$\{(z,w) : \ z \in D_r(0) \ \& \ |w - w_0| = \varepsilon\}$$

- We will build φ by Residue type Formula, using the circumference $C = \{|w - w_0| = \varepsilon\}$

First, we define $\eta : D_r(0) \longrightarrow \mathbb{C}$ as

$$\eta_{(z)} := \frac{1}{2\pi i} \oint_C \frac{\partial_w F(z,w)}{F(z,w)}\, dw = \#\text{ of soln of } F(z,T) \text{ in } |T - w_0| < \varepsilon$$

[<u>Reason</u>: if $P(t) = \prod_{j=1}^{\ell} (t - \alpha_j)^{n_j} \Rightarrow \frac{P'(t)}{P(t)} = \sum_{j=1}^{\ell} \frac{n_j}{t - \alpha_j}$ & $\frac{1}{2\pi i} \oint_C \frac{P'(t)}{P(t)}\, dt = \sum_{\substack{j=1 \\ \alpha_j \in \mathrm{Int}(C)}}^{\ell} n_j$]

By construction η is continuous and $\mathrm{im}(\eta) \subseteq \mathbb{Z}$, so it's constant

Since $\eta(0) = \mathrm{mult}(w_0, F_{(0,w)}) = 1$ by our choice of ε, we get $\boxed{\eta_{(z)} \equiv 1.}$

We define $\varphi : D(0,r) \longrightarrow \mathbb{C}$ via $\varphi_{(z)} := \frac{1}{2\pi i} \oint_C w \frac{\partial_w F(z,w)}{F(z,w)}\, dw$

<u>Claim</u>. $\varphi_{(z)}$ = the value w of the unique zero of $F(z,w)$ for $w \in D_\varepsilon(w_0)$.

Pf/ If $P(t) = \prod_{j=1}^{\ell} (t - \alpha_j)^{n_j} \Rightarrow \frac{1}{2\pi i} \oint_C \frac{t P'(t)}{P(t)}\, dt = \sum_{\substack{j=1 \\ \alpha_j \in \mathrm{Int}(C)}}^{\ell} n_j \alpha_j$

But since $\eta(z) = 1$, we know $F(z,w)$ only has 1 zero in $D_\varepsilon(w_0)$, so $\varphi(z)$ collects its value.

- $\varphi(0) = w_0$ by construction
- φ depends holomorphically on z so $\varphi \in \mathcal{O}(D_r(0))$.

The construction gives uniqueness for free! (The integral formula computes the solution by the Residue Theorem). ☐

<u>Next Time</u>: We'll see how to build (X, p, f) for $Y = \mathbb{P}^1$ & $Q_{(T)} = T^2 - g_{(z)}$
$\mathcal{M}(Y) = \mathbb{C}(z)$

§12.1 Proof of Uniqueness in Theorem 1 (§11.2)

To prove the uniqueness of the triple (X, p, F) from Theorem 1 we need the following technical result:

Proposition 1: Suppose we have 3 Riemann surfaces X, Y, Z and 2 proper n-sheeted holomorphic branched coverings $p: X \longrightarrow Y$ & $q: Z \longrightarrow Y$. Fix $B \subseteq Y$ closed, discrete & write $Y' = Y \setminus B$, $X' = X \setminus p^{-1}(B)$, $Z' := Z - q^{-1}(B)$.
Assume $p|_{X'}: X' \to Y'$ & $q|_{Z'}: Z' \to Y'$ are covering maps.
Then, every biholomorphism $\sigma': Z' \longrightarrow X'$ with $p \circ \sigma' = q|_{Z'}$ & $f \circ \sigma' = g|_{Z'}$ can be extended to a unit biholomorphism $\sigma: Z \longrightarrow X$ with $p \circ \sigma = q$ & $f \circ \sigma = g$.

Remark: In particular, this gives a bijection from $\mathrm{Deck}(X|Y)$ to $\mathrm{Deck}(X'|Y')$ given by restriction.

Definition: We say a branched holomorphic proper map $p: X \longrightarrow Y$ is galois if the associated covering map $p': X \setminus A \longrightarrow Y \setminus B$ (where $B = $ crit values(p) & $A = p^{-1}(B)$) is galois.

Proof of Proposition 1:

- Fix $b \in B$ & a coordinate nbhd (V, z) for b $\left(z: V \xrightarrow{\sim} \mathbb{D},\ b \longmapsto 0\right)$. Write $\boxed{V^*} = V \setminus \{b\}$. & assume V is small enough so that p & q are unbranched over V^* (ie $V^* \cap \{\text{crit pts of } p\} = V^* \cap \{\text{crit pts of } q\} = \phi$). We can do this because crit pts of proper non-const maps are discrete & closed.

- Fix $U_1, \ldots, U_m$ to be the connected components of $U := p^{-1}(V)$ $m \leq n$
- —— $W_1, \ldots, W_s$ —————————— $W := q^{-1}(V)$ $s \leq n$

Write $\boxed{U_i^*} := U_i \setminus p^{-1}(b)$ $i = 1, \ldots, m$ & $U^* = \bigsqcup_{i=1}^{m} U_i^*$

—— $\boxed{W_j^*} := W_j \setminus q^{-1}(b)$ $j = 1, \ldots, s$ & $W^* = \bigsqcup_{j=1}^{s} W_j^*$

<u>Claim</u>: $\sigma|_{W^*}: W^* \longrightarrow U^*$ is biholomorphic, so $m = s$.

Pf/ $p \circ \sigma = q$ so $\sigma|_{W^*}: W^* \longrightarrow U^*$. Since p & q are surjective, σ is.

Thus, $\sigma|_{W^*}$ is biholomorphic onto U^*.

- We reorder so that $\sigma|_{W_i^*}: W_i^* \longrightarrow U_i^*$ is biholomorphic.
- View $p|_{U_i^*}: U_i^* \longrightarrow V^*$ as a finite sheeted covering & $p|_{U_i}: U_i \to V$ is a branched covering, with $V^* \cong \mathbb{D}^*$ & $V \simeq \mathbb{D}$. By our classification of proper maps to $\mathbb{D}$, unbranched over $\mathbb{D}^*$, $U_i^* \underset{\text{biholo}}{\simeq} \mathbb{D}^*$ & $U_i \setminus U_i^* = \{1\,\text{pt}\}$. Call this pt a_i.
- Similarly, working with q we get $W_i \setminus W_i^* = \{c_i\}$ for a unique pt $c_i \in Z$

<u>Conclusion</u>: we can extend $\sigma|_{W_i^*}$ to W_i via $\sigma(c_i) = a_i$. $\forall i = 1..m$

By construction, $\sigma|_W: W \longrightarrow U$ is biholomorphic (extension is homeo & holo by Removable singularity Thm, just restrict to the corresponding map $W_i^* \longrightarrow U_i^*$) & extend by $b \longrightarrow b$ via $W_i^* \simeq \mathbb{D}^* \simeq U_i^*$)

- We do this extension for each $b \in B$ to get $\sigma: Z \longrightarrow X$ biholomorphic.
- By construction $p \circ \sigma = q$ holds & $f \circ \sigma = g$ follows from continuity. $\square$

<u>Proof of Uniqueness in Theorem 1</u>:

Uniqueness will follow from Proposition 1 & showing that after we remove the bad points to get top coverings & our hands are tied to define a restriction of σ

Pick another alg function (Z, q, g) & consider the discrete closed set

$$\boxed{B = q(\{\text{poles of } g\}) \cup \{\text{crit values of } q\} \subseteq Y}$$

Set $Y' = Y \setminus B$, $Z' = q^{-1}(Y')$, $X' = p^{-1}(Y') \subseteq |U|$

- Build $\sigma': Z' \longrightarrow X'$ biholomorphism compatible with p, q, f & g

Fix $z \in Z'$ & write $y := q(z) \in Y'$. Set $\varphi = q_*(\underbrace{\rho_z(g)}_{\mathcal{O}_{Z',z}}) \in \mathcal{O}_{Y',y}$ $(q: Z' \to Y'$ is local homeo$)$

<u>Claim 1</u>: $Q(\varphi) = 0$

Pf/ $\underset{\mathcal{M}(Y)[T]}{Q}(\varphi) = Q(q_*(\rho_z(g))) \underset{(**)}{=} q_*(\rho_z(q^*(Q)(g))) \underset{\text{def of alg function.}}{=} q_*(\rho_z(0)) = q_*(0) = 0$

(**) $[\ (T-a)(q_*(\rho_z(g))) = q_*(\rho_z(g)) - a = q_*(\rho_z(g) - q^*(a)) = q_*(\rho_z(q^*(T-a))(g))\ ]$

<u>Define</u>: $\sigma'(z) := \varphi. \in X'$ $\qquad \sigma' : Z' \longrightarrow X'$, $p_{|X'} \circ \sigma'(z) = y = q_{|Z'}(z)$

$f \circ \sigma'_{(z)} = g_{(z)}$ because $f(q_*(\rho_z(g))) = q_*(\rho_z(g))(y) \underset{\text{def } q_*}{=} \rho_z(g)(z) = g(z)$

<u>Claim 2</u>: σ' is continuous

Pf/ Pick $\mathcal{N}(U,s)$ open in $|\mathcal{O}|$ away from $p^{-1}(B)$, so $U \subseteq Y'$, $s \in \mathcal{O}(U)$

Then $(\sigma')^{-1}(\mathcal{N}(U,s)) = \{ z \in Z' \mid \sigma'(z) \in \mathcal{N}(U,s) \}$
$= \{ z \in Z' \mid q_*(\rho_z(g)) = [s]_y \text{ for } y = q(z) \in U \}$
$= q^{-1}(U)$ is open

Why? Since q is unbranched on Z', q is a local homeomorphism

Thus, given $y \in U$ & $z \in q^{-1}(y)$ we have $q: \begin{matrix} V' & \xrightarrow{\sim} & U' \subseteq U \\ \cup\!\shortmid & & \cup\!\shortmid \\ z & \longmapsto & y = q(z) \end{matrix}$ for suitable U', V'.

$\Longrightarrow$ $g_{|V'} = q^*(s|_{U'})$ says $(\sigma')^{-1}(\mathcal{N}(U',s)) = V' \ni z$

<u>Claim 3</u>: σ' is holomorphic & non-constant. (g is not constant)

Pf/ $p_{|X'} \circ \sigma' = q_{|Z'}$ & both $p_{|X'}$ & $q_{|Z'}$ are both holomorphic & local homeomorphisms.
So σ' is holomorphic

<u>Claim 4</u>: σ' is proper. ($\Rightarrow \sigma$ is surjective by Lemma 2 §5.2, because σ is non-cst, & X' is connected)

Pf/ The statement holds because $p_{|Y'}$ is continuous & $q_{|Z'}$ is proper. Indeed, if K is compact in $X' \Rightarrow p(K)$ is compact in Y'

$\Longrightarrow q^{-1}(p(K)) \subseteq X'$ is compact .

Now, we look at the diagram
$$\begin{tikzcd} \sigma'^{-1}(K) \subseteq Z' \arrow[rr, "\sigma'"] \arrow[dr, "q"'] & \arrow[d, phantom, "\circlearrowleft"] & X' \supseteq K \arrow[dl, "p"] \\ & Y' & \end{tikzcd}$$

To conclude: $\underbrace{\sigma'^{-1}(K)}_{\substack{\text{closed} \\ (X' \text{ is Hausdorff})}} \subseteq \underbrace{q^{-1}(p(K))}_{\text{compact}} \Rightarrow (\sigma')^{-1}(K)$ is compact.

<u>Claim 5</u>: σ is biholomorphic:

Since f & p both have the same degree, we conclude that $\deg(\sigma')=1$. Since σ' is a proper degree 1 holomorphism, we conclude it's biholomorphic onto its image. The surjectivity from claim 4 shows $\sigma'(Z')=X'$.

- To finish, we use Proposition 1 to extend $\sigma': Z' \to X'$ to $\sigma: Z \to X$ biholomorphically ensuring $p\circ\sigma = q$ & $f\circ\sigma = g$. We thus get $g=\sigma^* f$.

<u>Uniqueness of σ</u>: Any other $\tau: Z \to X$ with $g=\tau^* f$ will have to agree with σ. Otherwise $\tau(Z')=X'$ by construction, & $\alpha := \tau\circ\sigma^{-1}_{|X'}$ is a Deck transformation (since
$$\begin{tikzcd} X' \arrow[rr, "\tau\circ\sigma^{-1}_{|X'}"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowleft"] & X' \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$
)

Furthermore $\alpha^*(f)=f$ because $(\sigma^{-1}_{X'})^* = (\sigma_{|X'})_*$ & so
$$\alpha^* f = (\sigma^{-1}_{|X'})^* \circ \tau^* f = (\sigma^{-1}_{|X'})^*(g) = (\sigma^{-1}_{|X'})^*((\sigma_{|X'})^{-1})_*(f) = f$$

However, f takes distinct values on each of the n points in $p^{-1}(y)$ $\forall y\in Y$, so α must be id_X for the identity $\alpha^* f = f$ to hold.

§12.2. Algebraic functions and field extensions:

From Proposition 1 §12.1 we have a good notion of $\mathrm{Deck}(X|Y)$ whenever $p: X \longrightarrow Y$ is a proper degree n holomorphic map.

Furthermore, we have Φ $\mathrm{Deck}(X|Y) \hookrightarrow \mathrm{Aut}(\mathcal{M}(X))$ defined as

$$\sigma \longmapsto (f \longmapsto \sigma \cdot f := f \circ \sigma^{-1})$$

- Φ is a group homomorphism ($\mathrm{Deck}(X|Y) \circlearrowright \mathcal{M}(X)$.)
- The action fixes $p^*\mathcal{M}(Y) \subseteq \mathcal{M}(X)$ $\quad \left((p^*g)\circ\sigma^{-1} = g\circ p\circ\sigma^{-1} = g\circ p = p^*g \,.\right)$

Conclude: $\mathrm{Deck}(X|Y) \subseteq \mathrm{Gal}(\mathcal{M}(X)|p^*\mathcal{M}(Y))$

Our next theorem relates algebraic functions with the constructions from §11.3

Theorem 2: Given Y RS, $Q \subseteq \mathcal{M}(Y)[T]$ irred & monic, consider (X, p, f) the algebraic function defined by Q & the field extn $p^*: \underset{=K}{\mathcal{M}(Y)} \hookrightarrow \underbrace{p^*K}_{=L} \subseteq \mathcal{M}(X)$

Then: (1) $[L:K]=n$ & $L \simeq K[T]/(Q(T))$

(2) $\mathrm{Deck}(X|Y) := \{\sigma: X \xrightarrow[\text{holo}]{} X : p\circ\sigma = p\} \overset{\Phi}{\simeq} \mathrm{Gal}(L|K)$

(3) $p: X \longrightarrow Y$ is "galois" (ie $p: Y' \longrightarrow X'$ unbranched holo map is a galois cover) $\Longleftrightarrow$ $L|K$ is a Galois extension.

Proof (1) We know for any $y \in Y' = Y \setminus (\{\text{disc of } Q\} \cup \bigcup_{i=1}^{n} \{\text{Poles of } c_i\})$ f has exactly n distinct values on $p^{-1}(y)$. By the second part of Theorem 11.1, we see that $[L:K]=n$.

We have a ring homomorphism $K[T] \longrightarrow L$ $\quad$ Now $p^*Q(f) = 0$ by definition

$$R(T) \longmapsto p^*R(f)$$

Thus, the ring homomorphism factors through the ring hom $\frac{K[T]}{(Q)} \xrightarrow{\Psi} L$

The source is a field, so Ψ is injective. Both source and target are field ext. of K of degree n, so Ψ is an isomorphism.

(2) We know Φ: $\mathrm{Deck}(X/Y) \hookrightarrow \mathrm{Gal}(L|K)$ because $\sigma f \neq f$ $\forall \sigma \neq id_X$

(f takes n different values on $p^{-1}(y)$ for each $y \in Y' = Y \setminus (Z(\mathrm{disc}(Q)) \cup \{\text{poles of coeff of } Q\})$)

<u>Claim</u>: The map Φ is also surjective, so Φ is an iso.

Pf/ Pick $\alpha \in \mathrm{Gal}(L|K)$. Then $(X, p, \alpha \circ f)$ would also be an algebraic function defined by $Q(T)$ ($p^*Q(\alpha \circ f) = 0$ because $p^*Q(f) = 0$ & the Galois group permutes the roots of p^*Q).

By the uniqueness, we can find $\tau \in \mathrm{Deck}(X|Y)$ with $\alpha f = \tau^* f$

Take $\sigma = \tau^{-1}$ & notice $\sigma \cdot f = f \circ \sigma^{-1} = f \circ \tau = \tau^* f = \alpha f$.

Since $L = K(f)$, we see that $(f \longrightarrow \sigma f) \in \mathrm{Gal}(L|K)$ agrees with α

so $\alpha = \Phi(\sigma)$.

(3) Use the definition! $p: X \longrightarrow Y$ is Galois $\Leftrightarrow$ $\underbrace{p': X' \longrightarrow Y'}_{n\text{-sheeted covering}}$ is Galois $\Leftrightarrow$ $\mathrm{Deck}(X'|Y')$ has n elements

Similarly: $L|K$ is Galois if $\mathrm{Gal}(L|K)$ has size $[L:K] = n$.

Since $\mathrm{Deck}(X'|Y') \simeq \mathrm{Gal}(L|K)$ by (2), the statement holds $\square$