Lecture XIII: Examples of Algebraic functions & Puiseux Expansions

Fix Y a RS & $Q = T^n + c_1 T^{n-1} + \dots + c_n \in \mathcal{M}(Y)[T]$ irreducible

Definition: An algebraic function for Q & Y is a triple (X, p, f) when

(1) X is a RS

(2) $p : X \longrightarrow Y$ is a proper non-const holomorphic map of degree n ("n-sheeted branched covering")

(3) $f \in \mathcal{M}(X)$ with $\underbrace{(p^*Q)}_{\in \mathcal{M}(Y)[T]}(f) = 0$.

THM: The triple is unique up to ! biholomorphism: If (Z, q, g) is another such triple, then $\exists !\ \sigma : Z \longrightarrow X$ biholomorphism with

$$\begin{tikzcd} Z \arrow[rr, "\sigma"] \arrow[dr, "q"'] & \circlearrowleft & X \arrow[dl, "p"] \\ & Y & \end{tikzcd} \quad \& \quad \begin{tikzcd} Z \arrow[rr, "\sigma"] \arrow[dr, "g"'] & \circlearrowleft & X \arrow[dl, "f"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

§13.1 Examples

Set $Y = \mathbb{P}^1$ $\quad h(z) = (z - \alpha_1) \cdots (x - \alpha_m)$ polynomial with n distinct roots &

$$Q(T) := T^2 - h \in \mathcal{M}(\mathbb{P}^1)[T] = \mathbb{C}(z)[T]$$

Claim: $Q(T)$ is irreducible

Pf/ By contradiction, assume $\exists\ g \in \mathbb{C}(z)$ with $g^2 = h$ $\quad$ (soln to $T^2 = h$)

- Then, $g \in \mathcal{O}(\mathbb{C} \setminus \{\alpha_1, \dots, \alpha_m\})$ because $h \in \mathcal{O}(\mathbb{C})$
- By argument principle $I := \frac{1}{2\pi i} \int_{\odot_{\alpha_i}} \frac{g'(z)}{g(z)}\, dz \in \mathbb{Z}$ $\quad$ (# zeros of g in $\mathrm{Int}(\odot)$)
  ($\odot$ small enough)

But $\frac{g'(z)}{g(z)} = \frac{1}{2} \frac{h'}{h}$ $\quad$ so $\quad I = \frac{1}{2} \frac{1}{2\pi i} \oint_{\odot_{\alpha_i}} \frac{h'(z)}{h(z)} = \frac{1}{2} \cdot 1 = \frac{1}{2} \notin \mathbb{Z}$

Contr! □

$\Rightarrow$ Using Theorem §11.3, we have a RS X associated to Q + $\begin{array}{c} X \\ p \downarrow \\ \mathbb{P}^1 \end{array}$ 2-to-1 proper map

& $g \in \mathcal{M}(X)$ with $(p^*Q)(g) = 0$. $\quad$ (Write $g :=$ "$\sqrt{h}$")

Note: p is holomorphic & unbranched over $\mathbb{C} \setminus \{\alpha_1, \dots, \alpha_m\} = \mathbb{P}^1 \setminus B$ with

$B = \{\alpha_1, \dots, \alpha_m\} \cup \{\infty\}$

Q: • What do these surfaces look like? (They correspond to hyperelliptic curves)
• Where are the branch points of p?

• First, we look at $p|_{X'} : X' = X \setminus p^{-1}(B) \longrightarrow Y' = Y - B$ holomorphic 2-sheeted unbranched covering

$\Rightarrow$ given any $y \in Y'$ & $\varphi \in \mathcal{O}_{Y',y}$ with $\varphi^2 = \rho_y(h)$ we can do analytic cont^n on any path u in Y' starting from y

$$\begin{tikzcd}
 & \varphi \in X' \subseteq |\mathcal{O}_{Y'}| \subseteq |\mathcal{O}_{\mathbb{P}^1}| \arrow[d, "p"] \\
0 \in [0,1] \arrow[ur, "\exists!\,\hat{u}"] \arrow[r, "u"'] \arrow[ur, phantom, "\circlearrowleft"] & Y' \ni y'
\end{tikzcd}$$

$\hat{u}(0) = \varphi$

⚠ Y' is not simply connected so we cannot guarantee we have a section $s \in \mathcal{O}(Y')$ with $\rho_y(s) = \varphi$

• We have 2 solutions φ to $\varphi^2 = \rho_y(h)$, differing by sign. (2 points in X' over each $y \in Y'$)

Q: How many solutions do we have over each $b \in B$?

A: Only 1 over $\alpha_1, \dots, \alpha_m$. Over ∞, it depends on the parity of m

Lemma 1: X has exactly 1 point over each α_i ($\Rightarrow$ branch pts of p)

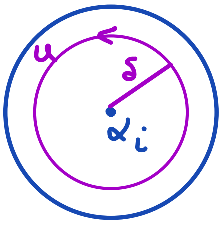
Pf/Idea: Isolate each a_i by a disc and look at the restriction of p on its preimage. The 2 connected components will have intersecting closures over $p^{-1}(a_i)$.

Given $i \in \{1, \dots, m\}$ fix $r_i > 0$ & $U_i = D_{r_i}(\alpha_i) = \{z \mid |z - a_i| < r_i\}$ with $U_i \cap B = \{\alpha_i\}$

• Consider $g(z) = \dfrac{h(z)}{z - \alpha_i} = \prod_{j \neq i} (z - \alpha_j)$

Note: g has no zeroes in U_i & U_i is simply connected. By Corollary §9.1 we can find 2 sections $G_1, G_2 : U_i \longrightarrow \mathbb{C}$ with $G_i^2 = g$ (solve for 1 germ & extend to a section)

Use $G = G_1$ to write $f(z) = (z-\alpha_i)\, G^2(z)$.



Fix $0<\delta<r_i$ & write $\xi \in$ as $\xi = a_i + \delta e^{i\theta}$ $\quad 0\le\theta\le 2\pi$

Set $\varphi \in \mathcal{O}_{Y',\xi}$ with $\varphi^2 = \rho_\xi(h_i)$

Then $\varphi(\xi) = \sqrt{\delta}\, e^{i\theta/2}\, G(\xi)$

By construction, $AC_u \varphi = -\varphi$

This forces $p^{-1}(U_i)$ to have a single connected component. Otherwise, our classification of proper holo maps onto $\mathbb{D}$, unbranched over $\mathbb{D}^*$ from Theorem 2 §8.1, would make each of the 2 connected components of $p^{-1}(U_i)$ biholomorphic to $\mathbb{D}$ & $AC_u \varphi = \varphi$ (path in $|\mathcal{O}|$ would not leave the corresponding component.) □

Lemma 2: The nature of $p^{-1}(\infty)$ depends on the parity of m. If m is odd, p is branched over ∞ (ie $p^{-1}(\infty) = \{y\}$), but if m is even, p is unbranched over ∞, so $p^{-1}(\infty)$ consists of 2 different pts.

Proof: We consider a nbhd U of ∞. Set $U^* = \{z : |z|>r\}$ for $r>>1$ so that $U\cap B = \emptyset$. Then $U = U^* \cup \{\infty\}$ is a nbhd of ∞ & $U \overset{\psi}{\simeq} \mathbb{D}_{1/r}(0) \simeq \mathbb{D}$
$z \longmapsto \frac{1}{z}$

Since $h(\infty) = \infty$ & $\nu(\infty, f) = m$, we can write

$$f(z) = z^m F(z)$$

where $F: \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ holomorphic & F has no zeroes in U.

So by Lemma 1 applied to $g = F\circ\psi^{-1}$ we see that F admits a square root in U

• Write $V^* = p^{-1}(U^*)$ & notice $p|_{V^*}: V^* \longrightarrow U^*$ is a 2-sheeted covering

We discuss two cases, depending on the parity of m.

CASE 1: m is odd

Using $\sqrt{F}$ on U we can find $g = z^{\frac{m-1}{2}}\sqrt{F} \in \mathcal{O}(U)$ with $h(z) = z(g(z))^2$.

Using the same AC argument ("$\alpha_i = \infty$") for $\sqrt{h}$ from Lemma 1 involving a sign change after looping around ∞ with a circle in U centered at 0 we see that $p^{-1}(\infty)$ has exactly 1 pt (V^* is connected)

CASE 2: m is even

We set $h(z) = H(z)^2$ for $H = z^{m/2}\sqrt{F}$. In this case, $\sqrt{h}$ has 2 distinct solutions: $\pm H(z)$ & each one will correspond to one connected comp of $\overline{V^*}$. Thus, we have 2 distinct pts in Y over ∞ & V^* has 2 comp $\square$

We see two concrete examples:

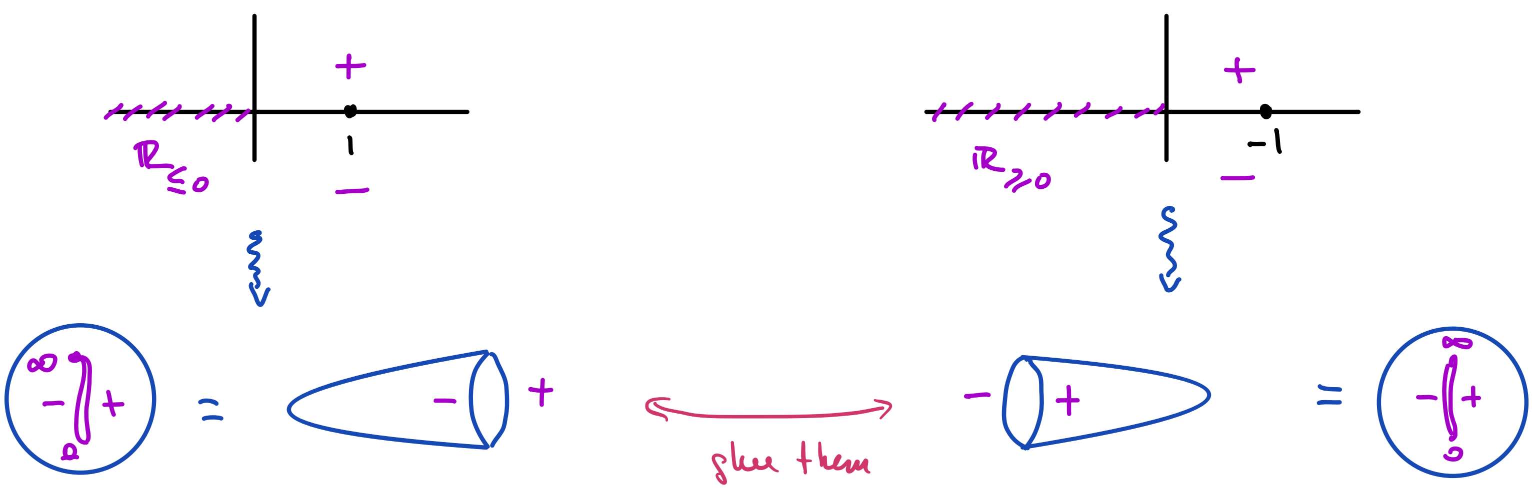
Example 1: $n=1$ odd & $f(z) = z \in \mathcal{M}(\mathbb{P}^1)$ $\qquad Q(z) = T^2 - z$

We have 2 square roots at $\mathcal{O}_{Y,1}$, one with value 1 at 1 : ______
-1 ______ :

n is odd, so p has 2 cut pts: 0 & ∞

Each germ can be extended to a holomorphic function in $\mathbb{C} \setminus$ cut

For φ^+: use $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ & for φ^- use $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$



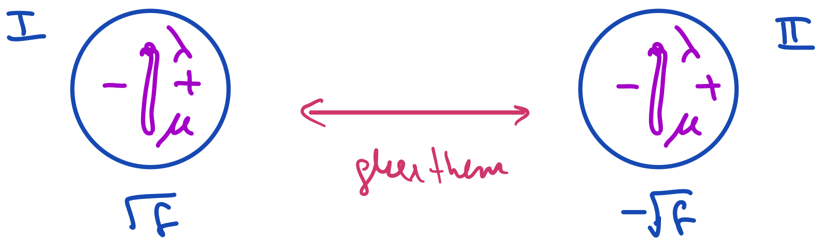
Do analytic continuation of φ^+ beyond the cut to get φ^-, so we can glue both sides along opposite signs, to get $Y = \mathbb{P}^1$ & $p: \underset{\mathbb{P}^1}{\overset{Y}{\downarrow}}$, $\underset{z}{\overset{(w,z)}{\downarrow}}$, $\underset{T^2 = f}{\overset{T}{\downarrow}}$

We have 1 choice of $\sqrt{z}$: $\Upsilon \longrightarrow \mathbb{C}$ meromorphic (single valued)

Example 2: $n=2$ then & $F(z) = (z-\lambda)(z-\mu)$

Now, we have 2 pts: λ, μ ; ∞ is not a crit pt for p.

We make a slit along the line segment joining λ & μ. We can pick 2 branches of $\sqrt{F}$ on the complement. So we have 2 choices

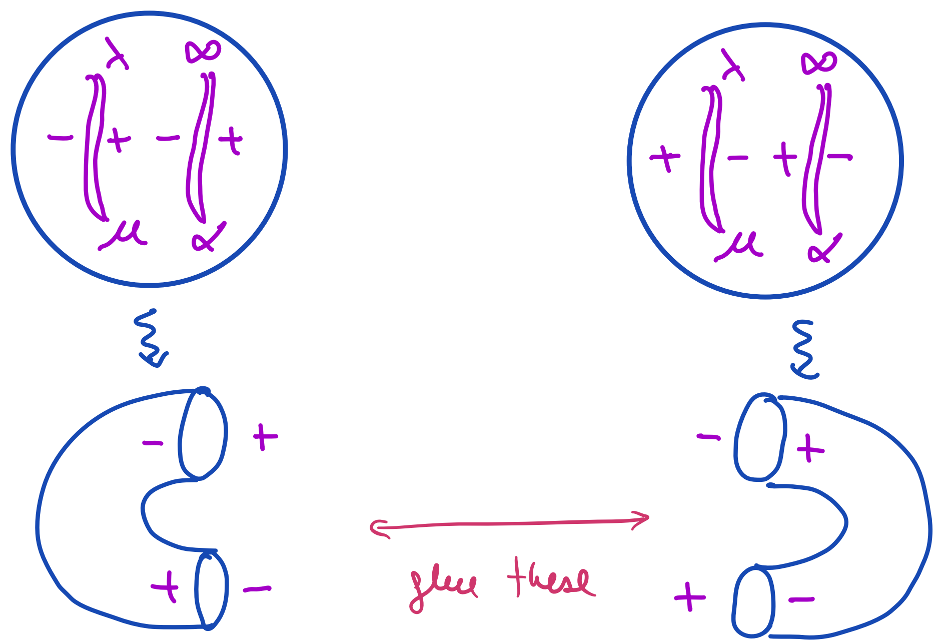


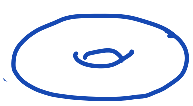
We glue them to get $\Upsilon = \mathbb{P}^1$

$$\begin{array}{ccc} \Upsilon = \mathbb{P}^1 & & z \\ p \downarrow & & \downarrow \\ \mathbb{P}^1 & & z \end{array}$$

Example 3: $n=3$ odd & $F(z) = (z-\lambda)(z-\mu)(z-\alpha)$

We have 4 critical points λ, μ, α & ∞. We make 2 cuts: one along the segment joining λ & μ & one along the line joining α to ∞. On the complement we can pick 2 branches of $\sqrt{F}$:

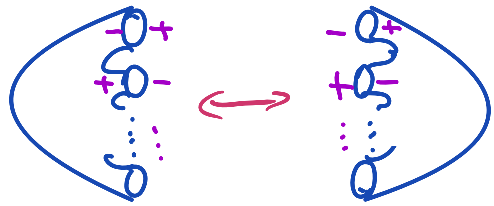


$\Rightarrow$ we get  so $\Upsilon = \mathbb{E}$

General construction: $h(x) = \prod_{i=1}^{n} (z-\alpha_i)$ we build $\lceil \frac{m}{2} \rceil$ slits

$\begin{cases} \bullet \text{ joining all pairs } \alpha_i, \alpha_{i+1} & \text{if } m \text{ even} \\ \bullet \text{ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ } & \text{for } i = \lceil \frac{m}{2} \rceil - 1 \text{ \& } \alpha_m \text{ to } \infty \text{ if } m \text{ odd)} \end{cases}$

We set Y by glueing



$\Rightarrow Y =$

surface of genus $\lceil \frac{m}{2} \rceil - 1$

Map p? $X = V(T^2 - h(z)) \subseteq \mathbb{P}^2$ & $p: \mathbb{P}^2 \to \mathbb{P}^1$, $(w, z) \mapsto z$

§13.2 Puiseux expansion:

We identify $\mathcal{M}_0$ = germs of meromorphic functions of $\mathbb{C}$ at $x=0$ with $\mathbb{C}\{\{z\}\} = \{ \sum_{\nu=-k}^{\infty} a_\nu z^\nu$ $k \in \mathbb{Z}$, $a_\nu \in \mathbb{C}$, converging in some $0 < |z| < r \}$

Pick $F(z,w) = w^n + c_1(z) w^{n-1} + \dots + c_n(z) \in \mathbb{C}\{\{z\}\}[w]$.

Theorem (Puiseux). Assume F is irreducible over $\mathbb{C}\{\{z\}\}$. Then, there exists a Laurent series $\varphi(\zeta) = \sum_{\nu=-k}^{\infty} a_\nu \zeta^\nu \in \mathbb{C}\{\{\zeta\}\}$ with $F(\zeta^n, \varphi(\zeta)) = 0$ viewed in $\mathbb{C}\{\{\zeta\}\}$. Furthermore if $c_i \in \mathbb{C}\{z\}$, then $\varphi \in \mathbb{C}\{z\}$ as well.

Equivalently: $F(z,w) = 0$ can be solved by viewing w as a series in $z^{1/n}$

Remark: There will be exactly n solutions ($w = \varphi(\varepsilon \zeta)$ where $\varepsilon^n = 1$.)

So $\mathbb{C}\{\{\zeta\}\}$ is the splitting field of $F \in \mathbb{C}\{\{z\}\}[w]$.

Proof of Theorem:

Lift c_i to $\mathcal{M}(D_r^*(0))$ & call the sections c_i. Since F is irred, we may assume the lift $F(z,w) \in \mathcal{M}(D_r(0))[w]$ is also irreducible. In addition, $F(0,w)$ has only simple roots, so 0 is away from the discriminant locus of F

- Pick $r > 0$ so that $\forall a \in D_r^*(0)$: $F(a,w)$ has only simple roots (we need to avoid the discriminant locus, but we can do this since 0 is not in it)

- Using Thm 1 §11.2, we consider the algebraic function (Y, p, f) defined by $F(z,w) \in \mathcal{O}(D_r(0))[w]$, so $p: Y \longrightarrow D_r(0)$ is a degree n proper holo map, unramified over $D_r^*(0)$. Set $Y' = Y - p^{-1}(0)$ (connected)

By our classification Thm 1 §8.1 $\exists\ \Psi: Y' \longrightarrow D^*_{r^{1/n}}(0)$ Ψ biholomorphic

$$\begin{tikzcd} Y' \arrow[r, "\Psi"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowleft"] & D^*_{r^{1/n}}(0) \arrow[dl, "\xi \mapsto \xi^n"] \\ D^*_r(0) & \end{tikzcd}$$

Furthermore $p^{-1}(0) = \{1\ \text{pt}\ \{=\{a\}$ & Ψ extends ! to a biholomorphism $\Psi: Y \to D^{(0)}_{r^{1/n}}$ with $\Psi(a) = 0$.

Write $\alpha = \Psi^{-1}: D^{(0)}_{r^{1/n}} \longrightarrow Y$. Then $p \circ \alpha\,(\xi) = \xi^n$

Since $F(p,f) = f^n_{(w)} + c_1 \circ p_{(w)} f^{n-1}_{(w)} + \cdots + c_n \circ p_{(w)} = 0$ in $\mathbb{C}[w]$, we set

$$F(\xi^n, f\circ\alpha(\xi)) = (f \circ \alpha(\xi))^n + c_1 \circ p(\alpha(\xi))(f\circ\alpha)^{n-1} + \cdots + c_n \circ p(\alpha(\xi))$$
$$= \alpha^*(F(p,f)) = \alpha^*(0) = 0.$$

Take $\varphi_{(\xi)} = f \circ \alpha_{(\xi)}$ to conclude. $\square$

Note: We can recover the power series expn of $f = \varphi \circ \alpha^{-1}$ if we can solve for φ directly & we can compute $\Psi = \alpha^{-1}$.

Summary: So far, we have built new RS from old ones from 3 perspectives:

① X RS $\rightsquigarrow$ $\tilde{X}$ univ cover & quotients by subgroups of $\mathrm{Deck}(\tilde{X}|X)$

② Y RS $\varphi \in \mathcal{O}_{Y,y}$ $\rightsquigarrow$ $X \subseteq |\mathcal{O}|$ conn comp of $|\mathcal{O}|$ containing φ (analytic cont.)

③ Y RS & $Q_{(T)} \subseteq \mathcal{M}(Y)(T)$ monic, irred $\rightsquigarrow$ (X, p, f) alg function

Missing construction: Build RS from differentiable 1-forms. (next lectures)