Lecture XIV: Differential forms

§14.1 Basic definitions:

Fix X RS & $U \subseteq X$ open chart with $z: U \xrightarrow{\sim} \mathbb{D}$ $\quad z = x+iy$

View X as a 2-dimensional real manifold (with local coord (x,y))

Definition: $\mathcal{E} :=$ sheaf of smooth (C^∞) functions on X (wrt the real structure) (of $\mathbb{C}$-algebras)

- Given $U \subseteq X$ open & $a \in X$, write (U', z) for open chart with $z: U' \xrightarrow{\sim} \mathbb{D}$, $a \longmapsto 0$

$\mathcal{E}(U) = \{ f: U \to \mathbb{C} \quad \forall a \in U$ & (U', z) open chart around a $\quad f_{|U'} \circ z^{-1}: \mathbb{D} \to \mathbb{C}$ is $C^\infty \}$

- Restriction maps are the natural ones.

- Given $a \in X$, $\mathcal{E}_a :=$ germs of C^∞-diff'ble functions at a.

- If (U,z) is a local chart, we define natural differential operators
$$\frac{\partial}{\partial x}\,;\ \frac{\partial}{\partial y} : \mathcal{E}(U) \longrightarrow \mathcal{E}(U)$$

by $\frac{\partial}{\partial x} f_{(a)} = \frac{\partial}{\partial x}(f \circ z^{-1})_{(z(a))}$ & $\frac{\partial}{\partial y} f_{(a)} = \frac{\partial}{\partial y}(f \circ z^{-1})_{(z(a))}$ for each $a \in U$

⚠ The definition of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ is chart dependent.

Definition: $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} : \mathcal{E}(U) \longrightarrow \mathcal{E}(U)$ via
$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Exercise: Show: $\mathcal{O}(U) = \operatorname{Ker}\left(\frac{\partial}{\partial \bar{z}}\right) \subseteq \mathcal{E}(U)$ (Cauchy Riemann: $f_y = i f_x$)

Definition: $\boxed{\mathfrak{m}_a} := \{ \eta \in \mathcal{E}_a : \eta(a) = 0 \}$ is a vector subspace of $\mathcal{E}_a$ (coord indep)

- We say $\eta \in \mathfrak{m}_a$ vanishes up to second order if $\frac{d\eta}{\partial x}(a) = \frac{d\eta}{dy}(a) = 0$ where η is lifted to (U,z) coordinate chart for a.

Set $\boxed{\mathfrak{m}_a^2} := \{ \eta \in \mathfrak{m}_a$ that vanish up to second order $\}$

Definition: $T_a^{(1)} := \mathfrak{m}_a / \mathfrak{m}_a^2$ is a vector space. It's called the cotangent space of X at a. This definition is coordinate independent!

If U open & $a \in U$ we build $\boxed{d_a : \mathcal{E}(U) \longrightarrow T_a^{(1)}}$ as $d_a f := \underbrace{(f - f(a))}_{\in \mathfrak{m}_a} \mod \mathfrak{m}_a^2 \quad \forall f$

Eg: $X = \mathbb{D}$ $\quad a = 0$ $\quad T_0^* = (x, y) \big/ (x,y)^2 = \dfrac{(x,y)}{(x^2, y^2, xy)}$

$dx := d_0 x = x \mod \mathfrak{m}_0^2$, $dy := d_0 y = y \mod \mathfrak{m}_0^2$

Remark 1: $d_a z := d_a(x + iy) = d_a x + i\, d_a y$

$d_a \bar{z} := d_a(x - iy) = d_a x - i\, d_a y$

Theorem 1: Given $a \in X$ & (U, z) coord chart for a, with $z = x + iy$, $T_a^{(1)}$ has basis $\{d_a x, d_a y\}$ Furthermore, if $\eta \in \mathcal{E}_a$ is represented by $f \in \mathcal{E}(U')$ $U' \subseteq U$

$\{d_a z, d_a \bar{z}\}$

we have $d_a f := \dfrac{\partial f}{\partial x}(a)\, d_a x + \dfrac{\partial f}{\partial y}(a)\, d_a y = \dfrac{\partial f}{\partial z}(a)\, d_a z + \dfrac{\partial f}{\partial \bar{z}}(a)\, d_a \bar{z}$

Proof: By Remark 1, we need only prove the statements for x & y

Claim 1: $\{d_a x, d_a y\}$ spans $T_a^{(1)}$.

Pf/ Fix $t \in T_a^{(1)}$ & write $\varphi \in \mathfrak{m}_a$ for a representative of t.

We can write down the Taylor series expansion of φ about a as

$$\varphi_{(x,y)} = \underbrace{\varphi(a)}_{=0} + c_1 x + c_2 y + \Psi \quad \text{with } c_1, c_2 \in \mathbb{C} \text{ \& } \Psi \in \mathfrak{m}_a^2.$$

In particular, $0 = c_1 x(a) + c_2 y(a) + 0$.

Regrouping $\varphi(x,y) = c_1 (\underbrace{x - x(a)}_{= d_a x \in \mathfrak{m}_a}) + c_2 (\underbrace{y - y(a)}_{= d_a y \in \mathfrak{m}_a}) + \Psi$, so $t = \overline{\varphi} \in \langle d_a x, d_a y \rangle$ in $T_a^{(1)}$

Claim 2: $\{d_a x, d_a y\}$ are l.i.

Pf/ If $0 = c_1 d_a x + c_2 d_a y \Rightarrow c_1 d_a x + c_2 d_a y \in \mathfrak{m}_a^2$

meaning $c_1(x - x_{(a)}) + c_2(y - y_{(a)}) \in \mathfrak{m}_a^2$

Take partial derivatives at a: $\left.\begin{array}{l} 0 = \frac{\partial}{\partial x}\Big|_a = c_1 \\ 0 = \frac{\partial}{\partial y}\Big|_a = c_2 \end{array}\right\}$ so $\{d_a x, d_a y\}$ is l.i.

Claim 3: The identity $d_a f := \frac{\partial f}{\partial x}(a)\, d_a x + \frac{\partial f}{\partial y}(a)\, d_a y$ holds.

Pf/ Pick $f \in \mathcal{E}(U')$ then $d_a f = f - f_{(a)}$ mod $\mathfrak{m}_a^2$

Using Taylor series expansion: $\underbrace{f_{(x,y)} - f_{(a)}}_{= d_a(f)} = \frac{\partial f}{\partial x}\Big|_{(a)} \underbrace{(x - x_{(a)})}_{= d_a x} + \frac{\partial f}{\partial y}\Big|_{(a)} \underbrace{(y - y_{(a)})}_{= d_a y} + g$

with $g \in \mathfrak{m}_a^2$

. To check the identity for $d_a f$ & the coordinates z & $\bar z$, we use the coord changes

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar z} \quad \& \quad \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar z}\right)$$

$$d_a x = \tfrac{1}{2}(d_a z + d_a \bar z) \quad \& \quad d_a y = \frac{1}{2i}(d_a z - d_a \bar z) = \frac{i}{2}(d_a \bar z - d_a z)$$

$$\text{So } d_a f = \left(\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar z}\right)_{(a)}\left(\tfrac{1}{2}(d_a z + d_a \bar z)\right) + i\left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar z}\right)_{(a)}\left(\frac{1}{2i}(d_a z - d_a \bar z)\right)$$

$$= \frac{\partial f}{\partial z}(a)\left(\tfrac{1}{2} d_a z + \tfrac{1}{2} d_a z\right) + \frac{\partial f}{\partial \bar z}(a)\left(\tfrac{1}{2} d_a \bar z + \tfrac{1}{2} d_a \bar z\right) +$$

$$+ \underbrace{\left(\tfrac{1}{2}\left(\frac{\partial f}{\partial z} d_a \bar z - \frac{\partial f}{\partial z} d_a \bar z\right)\right)}_{=0} + \underbrace{\tfrac{1}{2}\left(\frac{\partial f}{\partial \bar z}(a)\, d_a z - \frac{\partial f}{\partial \bar z}(a)\, d_a z\right)}_{=0}$$

□

Remark: The cotangent space is dual to the tangent space $= \left\langle \frac{d}{dx}, \frac{d}{dy} \right\rangle$ corresponding to infinitesimal paths through a

How? $\{$ Path $\gamma : (-\varepsilon, \varepsilon) \to X$, $\gamma(0) = a\} \times \mathcal{E}_a \longrightarrow \mathbb{C}$

$$(\gamma, f) \longmapsto \frac{d(f \circ \gamma)}{dt}\Big|_{t=0}$$

Up to shifting, we may assume $f_{(a)} = 0$. This map descends to $\{\text{paths}\} \times \frac{\mathfrak{m}_a}{\mathfrak{m}_a^2}$

§14.2 Differentiable 1- & 2-forms:

Next, we define 2 more sheaves by means of local coordinates:

Definition: $\mathcal{E}^{(1)}$ = sheaf of smooth 1-forms on X

Given U open & $a \in U$, write $(U'; z)$ for a local coord chart for a with $U' \subseteq U$

then $\omega \in \mathcal{E}^{1}(U')$ becomes $\omega = f\,dx + g\,dy$ with $f, g \in \mathcal{E}(U')$.

Lemma 1: The construction of $\mathcal{E}^{(1)}(U')$ depends on the choice of local coordinates but the local definitions glue to $\omega \in \mathcal{E}^{(1)}(U)$.

Pf/ Assume we have 2 local coordinates around $a \in U$.

(*) $\quad U' \xrightarrow[\varphi=(x,y)]{\sim} \mathbb{D} \quad \& \quad U' \xrightarrow[\psi=(u,v)]{\sim} \mathbb{D}$

Then we get $\quad \underset{(u,v)}{\mathbb{D}} \xrightarrow[\sim]{\varphi\circ\psi^{-1}} \underset{(x,y)}{\mathbb{D}}$, $\quad 0 \longmapsto 0$. In particular $\begin{array}{l} x = x(u,v) \\ y = y(u,v) \end{array}$

$$\omega = f\,dx + g\,dy = f\circ\varphi^{-1}(x(u,v), y(u,v))\left(\frac{dx}{du}du + \frac{dx}{dv}dv\right)$$
$$+\; g\circ\varphi^{-1}(x(u,v), y(u,v))\left(\frac{dy}{du}du + \frac{dy}{dv}dv\right)$$
$$= \underbrace{\left(f\circ\varphi^{-1}\circ\varphi\circ\psi^{-1}\frac{dx}{du} + g\circ\varphi\circ\psi^{-1}\frac{dy}{du}\right)}_{=F\circ\psi^{-1}(u,v)} du + \underbrace{\left(f\circ\varphi^{-1}\circ\varphi\circ\psi^{-1}\frac{dx}{dv} + g\circ\varphi\circ\psi^{-1}\frac{dy}{dv}\right)}_{=G\circ\psi^{-1}(u,v)} dv$$

This is the glueing data that allows us to build $\omega \in \mathcal{E}^{(1)}(U)$

Note: Alternative, we can write $\omega \in \mathcal{E}^{(1)}(U)$ via the coordinates dz & $d\bar{z}$:

$$\omega = \alpha\,dz + \beta\,d\bar{z} \quad \text{for } \alpha, \beta \in \mathcal{E}(U)$$

We use this to split $\mathcal{E}^{(1)}_a$ into 2 spaces $T_a^{1,0}$ & $T_a^{0,1}$ in a coord independent fashion. Note: we cannot use $d_a x$ & $d_a y$ to split $\mathcal{E}^{(1)}(U)$

Definition: $T_a^{1,0}$ = span of $d_a z$ in $\mathcal{E}_a^{(1)}$ & $T_a^{0,1}$ = span of $d_a\bar{z}$ in $\mathcal{E}_a^{(1)}$.

Lemma 2: The construction of $T_a^{1,0}$ & $T_a^{0,1}$ is independent of the local coordinate z picked around a.

Proof: We prove this by direct computation, setting $z' = u + iv$

Claim 1: $\frac{dz'}{dz}(a) = c \in \mathbb{C}^*$ & $\frac{d\bar{z}'}{d\bar{z}}(a) = \bar{c}$

Pf/ (*) $\mathbb{D} \xrightarrow{\varphi^{-1}} U \xrightarrow{\psi} \mathbb{D}$, $z \longmapsto z'$ is C^∞ & its inverse is also C^∞. Moreover, it's biholomorphic

Call $c := \frac{dz'}{dz}(a) = \frac{d\,\psi\circ\varphi^{-1}}{dz}(0) \implies$ it must be invertible.

$$\Rightarrow c = \frac{1}{2}\left(\frac{dz'}{dx}(0) - i\frac{dz'}{dy}(0)\right) = \frac{1}{2}\left(\frac{du}{dx}(0) + i\frac{dv}{dx}(0) - i\left(\frac{du}{dy}(0) + i\frac{dv}{dy}(0)\right)\right)$$

$$= \frac{1}{2}\left(\frac{du}{dx}(0) + \frac{dv}{dy}(0)\right) - \frac{1}{2}i\left(\frac{du}{dy}(0) - \frac{dv}{dx}(0)\right)$$

$\Rightarrow \bar{z}' = u(x,y) - i\,v(x,y)$ gives $\frac{d\bar{z}'}{dx}(a) = \frac{du}{dx}(0) - i\frac{dv}{dx}(0)$

$$\frac{d\bar{z}'}{dy}(a) = \frac{du}{dy}(0) - i\frac{dv}{dy}(0)$$

$$\Rightarrow \frac{d\bar{z}'}{d\bar{z}}(a) = \frac{1}{2}\left(\frac{d\bar{z}'}{dx}(a) + i\frac{d\bar{z}'}{dy}(a)\right) = \frac{1}{2}\left(\frac{du}{dx}(0) - i\frac{dv}{dx}(0) + i\left(\frac{du}{dy}(0) - i\frac{dv}{dy}(0)\right)\right)$$

$$= \frac{1}{2}\left(\frac{du}{dx}(0) + \frac{dv}{dy}(0)\right) + \frac{i}{2}\left(\frac{du}{dy}(0) - \frac{dv}{dx}(0)\right) = \bar{c}$$

Claim 2: $\frac{\partial z'}{\partial \bar{z}}(a) = \overline{\frac{\partial \bar{z}'}{\partial z}(a)} = 0$ (by Cauchy-Riemann)

Pf/ $$\frac{\partial z'}{\partial \bar{z}}(a) = \frac{1}{2}\left(\frac{\partial z'}{\partial x} + i\frac{dz'}{dy}\right)(a) = \frac{1}{2}\left(\frac{du}{dx}(0) + i\frac{dv}{dx}(0)\right) + \frac{i}{2}\left(\frac{du}{dy}(0) + i\frac{dv}{dy}(0)\right)$$

$$= \frac{1}{2}\left(\frac{du}{dx}(0) - \frac{dv}{dy}(0)\right) + \frac{1}{2}i\left(\frac{du}{dy}(0) + \frac{dv}{dx}(0)\right) = 0 \text{ by (CR)}$$

$$\frac{\partial \bar{z}'}{\partial z}(a) = \frac{1}{2}\left(\frac{\partial \bar{z}'}{\partial x} - i\frac{d\bar{z}'}{dy}\right)(a) = \frac{1}{2}\left(\frac{du}{dx}(0) - i\frac{dv}{dx}(0)\right) - \frac{i}{2}\left(\frac{du}{dy}(0) - i\frac{dv}{dy}(0)\right)$$

$$= \frac{1}{2}\left(\frac{du}{dx}(0) - \frac{dv}{dy}(0)\right) - \frac{i}{2}\left(\frac{du}{dy}(0) + \frac{dv}{dx}(0)\right)$$

Aside: CR equations for z' say $\frac{\partial z'}{\partial \bar z}=0$.

Conclusion: The change of basis $\{d_a z', d_a \bar z'\}$ to $\{d_a z, d_a \bar z\}$ is diagonal

$$d_a z' \underset{\downarrow}{=} \frac{\partial z'}{\partial z}(a)\, d_a z + \frac{\partial z'}{\partial \bar z}(a)\, d_a \bar z \underset{\downarrow}{=} c\, d_a z + 0\, d_a \bar z$$

Theorem § 14.1 Claims 1 & 2

$$d_a \bar z' = \frac{\partial \bar z'}{\partial z}(a)\, d_a z + \frac{\partial \bar z'}{\partial \bar z}(a)\, d_a \bar z = 0\, d_a z + \bar c\, d_a \bar z$$

Thus, $\mathbb{C}\, d_a z = \mathbb{C}\, d_a z'$ & $\mathbb{C}\, d_a \bar z = \mathbb{C}\, d_a \bar z'$ so $T_a^{1,0}$ & $T_a^{0,1}$ are defined in a coordinate independent way. $\square$

Corollary: Using $T_a^{(1)} = T_a^{(1,0)} \oplus T_a^{(0,1)}$ we define

$$d'_a : \mathcal{E}_a \longrightarrow T_a^{(1,0)} \quad \& \quad d''_a : \mathcal{E}_a \longrightarrow T_a^{(0,1)}$$

with $d_a = d'_a + d''_a$

Thus $d'_a(f) = \frac{\partial f}{\partial z}(a)\, d_a z$ & $d''_a(f) = \frac{\partial f}{\partial \bar z}(a)\, d_a \bar z$.

(Here, we lift $f \in \mathcal{E}_a$ to $f \in \mathcal{E}(U')$ with (U', z) coord nbhd of a.)

Lemma 2 says d'_a & d''_a are chart independent.

Alternatively $d'f = \frac{f_x - i f_y}{2}(dx + i dy)$

$d''f = \frac{f_x + i f_y}{2}(dx - i dy)$

Note: $f \in \mathcal{O}_a \subseteq \mathcal{E}_a \iff d''_a(f) = 0$

Definition: $\mathcal{E}^{(2)} :=$ sheaf of smooth 2-forms on X

For U open & $a \in U$, pick (U', z) a local coord chart for a

Then $w \in \mathcal{E}^{(2)}(U')$ if it can be written as $w = f\, dx \wedge dy$ for $f \in \mathcal{E}(U')$

In this case, the gluing data is given by the Jacobian relating the forms with respect to the 2 systems in (*)

$$f\, dx \wedge dy = g\, du \wedge dv \iff g\circ\varphi^{-1} = f\circ\varphi^{-1} \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}$$
$$= f\circ\varphi^{-1}\cdot\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)$$

§14.3 Exterior differentiation of forms:

• Next we use $\mathcal{E}$, $\mathcal{E}^{(1)}$ & $\mathcal{E}^{(2)}$ to build derivations $d, d', d'' : \mathcal{E}^{(1)} \longrightarrow \mathcal{E}^{(2)}$ & a chain complex: $\mathcal{E} \xrightarrow{d} \mathcal{E}^{(1)} \xrightarrow{d} \mathcal{E}^{(2)}$

① On $U \subseteq X$ open we can write $\omega \in \mathcal{E}^{(1)}(U)$ as a finite sum $\omega = \sum_k f_k\, dg_k$ where $f_k, g_k \in C^\infty$ (Eg: $\omega = f_1 dz + f_2 d\bar{z}$ if z is a local coord for $U' \subseteq U$) Set

$$d : \mathcal{E}^{(1)}(U) \longrightarrow \mathcal{E}^{(2)}(U) \quad ; \quad d' : \mathcal{E}^{(1)}(U) \longrightarrow \mathcal{E}^{(2)}(U) \quad ; \quad d'' : \mathcal{E}^{(1)}(U) \longrightarrow \mathcal{E}^{(2)}(U)$$
$$\omega \longmapsto \sum_k df_k \wedge dg_k \qquad \omega \longmapsto \sum_k d'f_k \wedge dg_k \qquad \omega \longrightarrow \sum d''f_k \wedge dg_k$$

Lemma 1: d, d', d'' are well defined (ie indep of the expression for ω)

Proof: We do it for d (The other two are similar)

Assume $\omega = \sum_k f_k\, dg_k = \sum_j \tilde{f}_j\, d\tilde{g}_j$ & work on a coord nbhd (U, z) where $z = x+iy$.

We want to show $\sum_k df_k \wedge dg_k = \sum_j d\tilde{f}_j \wedge d\tilde{g}_j$.

Note: $dg_k = \frac{\partial g_k}{\partial x} dx + \frac{\partial g_k}{\partial y} dy$
$$d\tilde{g}_j = \frac{\partial \tilde{g}_j}{\partial x} dx + \frac{\partial \tilde{g}_j}{\partial y} dy$$

Then, from the expressions of ω, we get

$$(1) = \sum_k f_k \frac{\partial g_k}{\partial x} = \sum_j \tilde{f}_j \frac{\partial \tilde{g}_j}{\partial x}$$
$$(2) = \sum f_k \frac{\partial g_k}{\partial y} = \sum_j \tilde{f}_j \frac{\partial \tilde{g}_k}{\partial y}$$

($\{dx, dy\}$ is a basis for $\mathcal{E}^{(1)}_{(U)}$)

Next, compute $\frac{\partial}{\partial y}(1) - \frac{\partial}{\partial x}(2)$ using 2 expressions

$$\frac{\partial}{\partial y}(1) - \frac{\partial}{\partial x}(2) = \sum_k \left(\frac{\partial f_k}{\partial y} \frac{\partial g_k}{\partial x} + f_k \frac{\partial^2 g_k}{\partial y \partial x} \right) - \sum_k \left(\frac{\partial f_k}{\partial x} \frac{\partial g_k}{\partial y} + f_k \frac{\partial^2 g_k}{\partial x \partial y} \right)$$

$$= \sum_k \left(\frac{\partial f_k}{\partial y} \frac{\partial g_k}{\partial x} - \frac{\partial f_k}{\partial x} \frac{\partial g_k}{\partial y} \right)$$

$$= \sum_j \left(\frac{\partial \tilde{f}_j}{\partial y} \frac{\partial \tilde{g}_j}{\partial x} - \frac{\partial \tilde{f}_j}{\partial x} \frac{\partial \tilde{g}_j}{\partial y} \right) \qquad (**)$$

Now

$$df_k \wedge dg_k = \left(\frac{\partial f_k}{\partial x} dx + \frac{\partial f_k}{\partial y} dy \right) \wedge dg_k$$

$$= \left(\frac{\partial f_k}{\partial x} dx + \frac{\partial f_k}{\partial y} dy \right) \wedge \left(\frac{\partial g_k}{\partial x} dx + \frac{\partial g_k}{\partial y} dy \right)$$

$$= \left(\frac{\partial f_k}{\partial x} \frac{\partial g_k}{\partial y} - \frac{\partial f_k}{\partial y} \frac{\partial g_k}{\partial x} \right) dx \wedge dy$$

Similarly, $d\tilde{f}_j \wedge d\tilde{g}_j = \left(\frac{\partial \tilde{f}_j}{\partial x} \frac{\partial \tilde{g}_j}{\partial y} - \frac{\partial \tilde{f}_j}{\partial y} \frac{\partial \tilde{g}_j}{\partial x} \right) dx \wedge dy$

Conclude:

$$\sum_k df_k \wedge dg_k = \sum_k \left(\frac{\partial f_k}{\partial y} \frac{\partial g_k}{\partial x} - \frac{\partial f_k}{\partial x} \frac{\partial g_k}{\partial y} \right) dx \wedge dy$$

$$\underset{(**)}{=} \sum_j \left(\frac{\partial \tilde{f}_j}{\partial y} \frac{\partial \tilde{g}_j}{\partial x} - \frac{\partial \tilde{f}_j}{\partial x} \frac{\partial \tilde{g}_j}{\partial y} \right) dx \wedge dy = \sum_j d\tilde{f}_j \wedge d\tilde{g}_j$$

So the definition of $d\omega$ is independent of the expression. $\square$

Properties: Given U open subset of X RS, $f \in \mathcal{E}(U)$ & $\omega \in \mathcal{E}^{(1)}(U)$, we have

(1) $ddf = d'd'f = d''d''f = 0$

(2) $d\omega = d'\omega + d''\omega$

(3) $d(f\omega) = df \wedge \omega + f\, d\omega$ [Leibniz Rule]

$d'(f\omega) = d'f \wedge \omega + f\, d'\omega$

$d''(f\omega) = d''f \wedge \omega + f\, d''\omega$

Proof (1) $ddf = d(1 \cdot df) = d1 \wedge df = 0 \wedge df = 0$

$$d'd'f = d'\left(\frac{\partial f}{\partial z}\,dz\right) = d'\left(\frac{\partial f}{\partial z}\right)\wedge dz = \frac{\partial^2 f}{\partial^2 z}\,dz\wedge dz = 0$$

$$d''d''f = d''\left(\frac{\partial f}{\partial \bar z}\,d\bar z\right) = d''\left(\frac{\partial f}{\partial z}\right)\wedge d\bar z = \frac{\partial^2 f}{\partial \bar z\,\partial z}\,d\bar z\wedge d\bar z = 0$$

(2) $d = d' + d''$ on $\mathcal{E}_{(U)} \longrightarrow \mathcal{E}^{(1)}_{(U)}$

So $d = d' + d''$ on $\mathcal{E}^{(1)}_{(U)} \longrightarrow \mathcal{E}^{(2)}_{(U)}$ since

$$d\Big(\sum_k f_k\,dg_k\Big) = \sum_k df_k\wedge dg_k$$

$$d'\Big(\sum_k f_k\,dg_k\Big) = \sum_k d'f_k\wedge dg_k$$

$$d''\Big(\sum_k f_k\,dg_k\Big) = \sum_k \underbrace{d''f_k}_{\in\,\mathcal{E}^{(1)}_{(U)}}\wedge dg_k$$

(3) Write $\omega = \sum h_k\,dg_k$ & use $\underset{d'}{d}(ab) = a\,\underset{d'}{d}b + b\,\underset{d'}{d}a$

$$d\Big(f\sum_k h_k\,dg_k\Big) = d\Big(\sum_k (fh_k)\,dg_k\Big) = \sum_k d(fh_k)\wedge dg_k$$

$$\underset{\text{Prod Rule}}{=} \sum_k f\,dh_k\wedge dg_k + \sum_k (h_k\,df)\wedge dg_k$$

$$\underset{\wedge \text{ is } \mathcal{E}(U)\text{-linear}}{=} \underbrace{f\sum_k dh_k\wedge dg_k}_{=\,d\omega} + df\wedge\underbrace{\Big(\sum_k h_k\,dg_k\Big)}_{=\,\omega}$$

$$d'\Big(f\sum_k h_k\,dg_k\Big) = d'\Big(\sum_k (fh_k)\,dg_k\Big) = \sum_k d'(fh_k)\wedge dg_k$$

$$= \sum_k f(d'h_k)\wedge dg_k + \sum_k (d'f\,h_k)\wedge dg_k$$

$$= f\,d'(\omega) + d'f\wedge\omega$$

$d'' = d - d'$ so we get $d''(f\omega) = f\,d''(\omega) + d''f\wedge\omega$ $\square$

② We can use the derivation d to build a chain complex.

On (U,z) local chart:

$$\mathcal{E}(U) \xrightarrow{\ d\ } \mathcal{E}^{(1)}(U) \xrightarrow{\ d\ } \mathcal{E}^{(2)}(U)$$

$$f \longmapsto f_x\,dx + f_y\,dy$$

$$a\,dx + b\,dy \longmapsto (-a_y + b_x)\,dx\wedge dy$$

More precisely $d(a\,dx + b\,dy) = da \wedge dx + db \wedge dy$

$$= (a_x dx + a_y dy) \wedge dx + (b_x dx + b_y dy) \wedge dy$$

$$= a_x \underbrace{dx \wedge dx}_{=0} + a_y \underbrace{dy \wedge dx}_{= -d_x \wedge dy} + b_x dx \wedge dy + b_y \underbrace{dy \wedge dy}_{=0}$$

These glue to give $\mathcal{E} \xrightarrow{d} \mathcal{E}^{(1)} \xrightarrow{d} \mathcal{E}^{(2)}$

We know $d^2 = 0$ by Property 1 on the previous page.

<u>Lemma 2</u>: Using $d'' : \mathcal{E}_{(U)} \longrightarrow \mathcal{E}^{(0,1)}_{(U)} \subseteq \mathcal{E}^{(1)}_{(U)}$ & $d' : \mathcal{E}^{(1)}_{(U)} \longrightarrow \mathcal{E}^{(2)}_{(U)}$ on (U, z) local chart we get $d'd''(f) = \dfrac{\partial^2 f}{\partial z \partial \bar{z}} dz \wedge d\bar{z} = \dfrac{1}{2i} \Big(\underbrace{\dfrac{\partial^2 f}{\partial^2 x} + \dfrac{\partial^2 f}{\partial^2 y}}_{= \Delta(f)} \Big) dx \wedge dy.$

<u>Definition:</u> We say $f \in \mathcal{E}(U)$ is <u>harmonic</u> if $d'd''(f) = 0$ (ie $\Delta(f) = 0$)

§14.4 <u>Pull-back of differential forms</u>:

Find $F : X \longrightarrow Y$ holomorphic map between two Riemann surfaces.

Fix $U \subseteq Y$ open & consider $F^* : \mathcal{E}_Y(U) \longrightarrow \mathcal{E}_X(F^{-1}(U))$

$$f \longmapsto f \circ F$$

We get an induced map on smooth k-forms $(k = 1, 2)$

$$F^* : \mathcal{E}^{(k)}_Y(U) \longrightarrow \mathcal{E}^{(k)}_X(F^{-1}(U))$$

We get the following expressions:

For $k=1$ $\omega = \sum_k f_k\, dg_k \longmapsto \sum_k f_k \circ F\, d(g_k \circ F) = \sum_k F^*(f_k)\, d(F^* g_k)$

$(f_k, g_k \in \mathcal{E}_Y(U))$

For $k=2$ $\omega = \sum_k f_k\, dg_k \wedge dh_k \longmapsto \sum_k (f_k \circ F)\, d(g_k \circ F) \wedge d(h_k \circ F)$

$(f_k, g_k, h_k \in \mathcal{E}_Y(U))$ $= \sum_k F^* f_k\, d(F^* g_k) \wedge d(F^* h_k)$.

Need to show this is independent of the expression for ω.

(Alternative, work with local coordinates & show they glue to give F^*)

Proposition: F^* commutes with d, d' & d''

Proof: It's enough to work in local coordinates. Pick (U,z) local chart in Y.

Pick (V,z') local chart with $V \subseteq F^{-1}(U)$

- Set $k=0$ & $g \in \mathcal{E}(U) \Rightarrow dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy$

$$F^*(dg) = F^*(1\,dg) = F^*(1)\,d(F^*g) = 1\,d(F^*g) = d(F^*g)$$

In coordinates:

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[d, "u+iv=w"'] & U \arrow[d, "z=x+iy"] \arrow[dr, bend left, "g"] & \\ \mathbb{D} \arrow[r] & \mathbb{D} & \mathbb{C} \end{tikzcd}$$

$$d(\underbrace{F^*}_{\in \mathcal{E}_x(F^{-1}(U))}) = \frac{d(g\circ F\circ w^{-1})}{\partial u}du + \frac{\partial(g\circ F\circ w^{-1})}{\partial v}dv$$

$$F^*(dg) = F^*\left(\frac{\partial(g\circ z^{-1})}{\partial x}dx + \frac{\partial(g\circ z^{-1})}{\partial y}dy\right) = \frac{\partial(g\circ z^{-1})}{\partial x}\Big|_{z\circ F\circ w^{-1}} d(\overset{x\circ F}{F^*x}) + \frac{\partial(g\circ z^{-1})}{\partial y}\Big|_{z\circ F\circ w^{-1}} d(F^*y)$$

But,
$$d(F^*x) = \frac{\partial\, x\circ F\circ w^{-1}}{\partial u}du + \frac{\partial\, x\circ F\circ w^{-1}}{\partial v}dv$$
$$d(F^*y) = \frac{\partial\, y\circ F\circ w^{-1}}{\partial u}du + \frac{\partial\, y\circ F\circ w^{-1}}{\partial v}dv$$

To show $F^*(dg) = dF^*g$ on $\mathbb{D} \underset{w^{-1}}{\cong} V$, we need to confirm

$$\frac{\partial}{\partial u}(g\circ F\circ w^{-1}) = \frac{\partial(g\circ z^{-1})}{\partial x}\frac{\partial(x\circ F\circ w^{-1})}{\partial u} + \frac{\partial\, g\circ z^{-1}}{\partial y}\frac{\partial\, y\circ F\circ w^{-1}}{\partial u}$$

& similarly for $\frac{\partial}{\partial v}$.

But this is precisely the chain rule using $g\circ F\circ w^{-1} = (g\circ z^{-1})\circ(z\circ F\circ w^{-1})$ & $z=(x,y)$

Next, we check the statements for d' & d'': Since $d = d' + d''$, it's enough to check d'

Ideally

$$\begin{array}{ccc} \mathcal{E}(V) & \overset{d'}{\underset{d''}{\rightrightarrows}} & \mathcal{E}^{(1,0)}(V) \oplus \mathcal{E}^{(0,1)}(V) \\ F^*\uparrow & \circlearrowleft & \uparrow F^* \\ \mathcal{E}(U) & \overset{d'}{\underset{d''}{\rightrightarrows}} & \mathcal{E}^{(1,0)}(U) \oplus \mathcal{E}^{(0,1)}(V) \end{array}$$

If the split structures are respected, then F^* commutes with d' & d''

- $F^*(d'g) = F^*(g_z\, dz) = F^*(g_z)\, d(F^*z) = g_z \circ F\, d(F^*z)$

$$d(F^*z) = \frac{\partial(z\circ F)}{\partial w}dw + \frac{\partial(z\circ F)}{\partial \bar{w}}d\bar{w}$$

Since F is holomorphic, we have that $z\circ F$ is also holomorphic, thus

$$\frac{\partial z\circ F}{\partial \bar{w}} = 0$$

Conclude: $F^*(d'g) = g_z\circ F\ \frac{\partial(z\circ F)}{\partial w}dw$

- $d'(F^*g) = (F^*g)_w\, dw = (g\circ F)_w\, dw$

$$(g\circ F)_w = (g\circ z^{-1})\circ(z\circ F)_w = \frac{\partial(g\circ z^{-1})}{\partial z}(z\circ F)\cdot\frac{\partial(z\circ F)}{\partial w}$$

$$= g_z(F)\cdot\frac{\partial(z\circ F)}{\partial w}$$

So we get the equality we wanted

- Set $k=1$ & $\omega = \sum_k df_k \wedge dg_k \in \mathcal{E}^{(1)}_Y(U)$

$$F^*(d\omega) = F^*\Big(\sum_k df_k\wedge dg_k\Big) = \sum_k d(F^*f_k)\wedge dF^*g_k$$
$$= d\Big(\sum_k F^*f_k \wedge dF^*g_k\Big) = d(F^*(\omega))$$

$$F^*(d'\omega) = F^*\Big(\sum_k d'f_k\wedge dg_k\Big) = \sum_k d'(F^*f_k)\wedge dF^*g_k$$
$$= d'\Big(\sum_k F^*f_k \wedge dF^*g_k\Big) = d'(F^*(\omega))$$

$$F^*(d''\omega) = F^*\Big(\sum_k d''f_k\wedge dg_k\Big) = \sum_k d''(F^*f_k)\wedge dF^*g_k$$
$$= d''\Big(\sum_k F^*f_k \wedge dF^*g_k\Big) = d''(F^*(\omega))$$

Consequence: If $f\in\mathcal{E}(U)$ is harmonic, then $F^*_0 f = f\circ F \in \mathcal{E}(F^{-1}(U))$ is also harmonic

PF/ $d'd''(F^*f) = d'F^*(d''f) = F^*(d'd''f) = F^*(0) = 0$ $\square$