Lecture XV: Integration of differential 1-forms

Last Time, we defined differential forms on RS by local coordinates.

g C^∞ function on $U \underset{z=x+iy}{\simeq} \mathbb{D}$ $\quad dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = \frac{\partial g}{\partial z}dz + \frac{\partial g}{\partial \bar{z}}d\bar{z}$

$U \subseteq X$ open $\quad \mathcal{E}(U) = \{$ smooth C^∞-diff'ble functions on $U\}$

$\mathcal{E}^{(1)}(U) = $ 1-forms $= \{\sum_{k \text{ finite}} f_k\, dg_k \quad f_k, g_k \in \mathcal{E}(U)\}$

$\mathcal{E}^{(2)}(U) = $ 2-forms $= \{\sum_{k \text{ finite}} f_k\, dh_k \wedge dg_k \ : \ f_k, h_k, g_k \in \mathcal{E}(U)\}$

Chain complex: $\quad \mathcal{E}(U) \xrightarrow{d} \mathcal{E}^{(1)}(U) \longrightarrow \mathcal{E}^{(2)}(U)$

$$\omega = \sum_k f_k\, dg_k \longmapsto \sum_k df_k \wedge dg_k$$

$F: X \longrightarrow Y$ holomorphic $\rightsquigarrow F^*: \mathcal{E}_Y^{(k)}(U) \longrightarrow \mathcal{E}_X^{(k)}(F^{-1}(U)) \qquad k=0,1,2$

- $k=0$ $\quad F^*(f) = f \circ F$
- $k=1$ $\quad F^*\left(\sum_j f_j\, dg_j\right) = \sum_j F^*(f_j)\, d(F^* g_j)$
- $k=2$ $\quad F^*\left(\sum_j f_j\, dh_j \wedge dg_j\right) = \sum_j F^*(f_j)\, d(F^* h_j) \wedge d(F^* g_j)$

KEY $\quad F^*$ commutes with d $\qquad$ (we'll use this today!)

TODAY: Discuss integration of differential 1-forms & periods of closed 1-forms

§15.1 Integration of 1-forms:

Fix X a RS, $\omega \in \mathcal{E}^{(1)}(X)$ &

Definition: We say a curve $\gamma: [0,1] \longrightarrow X$ is piecewise differentiable if we can find ① a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$
& ② coordinate charts (U_k, z_k) with $U_k \xrightarrow[z_k]{\sim} \mathbb{D}$ & $z_k = x_k + i y_k$ for $k=1,\dots,n$
with $\gamma([t_{k-1}, t_k]) \subseteq U_k$ & $x_k \circ \gamma: [t_{k-1}, t_k] \longrightarrow \mathbb{R}$, $y_k \circ \gamma: [t_{k-1}, t_k] \longrightarrow \mathbb{R}$ are C^1-differentiable

Definition: The path integral of the 1-form ω along γ is defined via the formula

$$\int_\gamma \omega := \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left(f_k(\gamma(t)) \frac{d(x_k\circ\gamma)}{dt} + g_k(\gamma(t)) \frac{d(y_k\circ\gamma)}{dt} \right) dt$$

if $\omega|_{U_k} = f_k\, dx_k + g_k\, dy_k$.

Lemma 1: $\int_\gamma \omega$ is well-defined (ie, independent of the local charts & the partition)

Lemma 2: $\int_\gamma \omega$ is independent of the choice of parameterization of γ.

Pf/ Write $\sigma(t) = \gamma\circ\tau(t)$ for $\tau : [0,1] \to [0,1]$ C^∞ & invertible. & use the chain rule to show $\int_\gamma \omega = \int_\sigma \omega$.

Note: the construction ensures that $\int_\gamma \omega = \sum_{k=1}^{n} \int_{\gamma_k} \omega$ where $\gamma_k = \gamma|_{[t_{k-1},t_k]}$ $\forall k=1\cdots n$

Theorem: If X is a RS, $F\in \mathcal{E}(X)$ & $\gamma : [0,1]\to X$ is piecewise diff'ble, then

$$\int_\gamma dF = F(\gamma(1)) - F(\gamma(0))$$

Pf/ Do it locally (assume X is a local chart & γ is differentiable) & add up the results.

The local case is the Fundamental Theorem of Calculus. Indeed, in (U_k, z_k)

$$dF = \frac{\partial F}{\partial x_k} dx_k + \frac{\partial F}{\partial y_k} dy_k$$

so $$\int_{\gamma_k} dF = \int_{t_{k-1}}^{t_k} \left(\frac{\partial F}{\partial x_k}(\gamma(t)) \frac{\partial(x_k\circ\gamma)}{\partial t} + \frac{\partial F}{\partial y_k}(\gamma(t)) \frac{\partial(y_k\circ\gamma)}{dt} \right) dt$$

$$= \int_{t_{k-1}}^{t_k} \frac{\partial F\circ\gamma}{\partial t}\, dt \underset{\text{FTC}}{=} F\circ\gamma(t_k) - F\circ\gamma(t_{k-1})$$

$$\Rightarrow \int_{\gamma_k} dF = \sum_{k=1}^{n} \int_{\gamma_k} dF = \sum_{k=1}^{n} \left(F\circ\gamma(t_k) - F\circ\gamma(t_{k-1}) \right) = F(\gamma(1)) - F(\gamma(0))$$

§15.2 Primitive Forms:

Definition: Given X a RS & $\omega \in \mathcal{E}^{(1)}(X)$. We say ω is exact if $\exists\, F \in \mathcal{E}(X)$ with $\omega = dF$. We call F a primitive for ω.

Lemma 1: F is unique up to a constant

Pf/ It is enough to show $dF = 0$ forces $F = 0$. This follows from the Theorem above. We can show F is locally constant (on $U \simeq \mathbb{D}$ we have $F(0) = F(z)$ $\forall z \in \mathbb{D}$ because $0 = \int_{[0,z]} \overset{=0}{dF} = F(z) - F(0)$) Since X is connected & F is continuous, it will be constant on X. ☐

Lemma 2: Any exact form ω is closed, ie $d\omega = 0$ (use $d^2 = 0$)

Note: If $\omega \in \mathcal{E}^{(1)}(X)$ has a primitive, then $\int_\gamma \omega$ only depends on $\gamma(0)$ & $\gamma(1)$

- As expected, not every closed form is exact, but primitives can be constructed locally. The issue arises when trying to glue these local versions.

For local purposes, we can assume $X = D_r(0) \subseteq \mathbb{C}$

Proposition 1: Any $\omega \in \mathcal{E}^{(1)}(D_r(0))$ closed is exact.

Proof Write $\omega = f\,dx + g\,dy$ Since ω is closed, we have $d\omega = 0$

Thus $\underset{\mathcal{E}^{(2)}(D_r(0))}{\underset{\cap}{d\omega}} = df \wedge dx + dg \wedge dy = (-f_y + g_x)\, dx \wedge dy = 0$ forcing $f_y = g_x$ (consistency condition)

Any $F \in \mathcal{E}(D_r(0))$ with $dF = \omega$ will satisfy $F_x = f$ & $F_y = g$.

The closed condition says $F_{xy} = F_{yx}$, as expected for C^∞ functions. We build F using integrals along radial paths starting from $(0,0)$. $\gamma_z : [0,1] \to D_r(0)$, $t \mapsto tz$

- Set $\boxed{F(z)} := \int_{\gamma_z} \omega = \int_{\gamma_z} f\,dx + g\,dy = \int_0^1 \big(f(tx,ty)\,x + g(tx,ty)\,y\big)\,dt$
  $z = (x,y)$

$F(z) \in C^\infty$ by construction. Need to confirm $dF = \omega$

$$\frac{\partial F}{\partial x} = \int_0^1 \frac{\partial}{\partial x}\left(f(tx,ty)\,x + g(tx,ty)\,y\right) dt$$

$$= \int_0^1 \Big(\frac{\partial f}{\partial x}(tx,ty)\,t\,x + f(tx,ty) + \underbrace{\frac{\partial g}{\partial x}(tx,ty)}_{\frac{\partial f}{\partial y}(tx,ty)} \cdot t\,y \Big)\, dt$$

$$= \int_0^1 \left(t\left(\frac{\partial f(tx,ty)}{\partial t}\right) + f(tx,ty)\right) dt = \int_0^1 \frac{\partial}{\partial t}\left(t\, f(tx,ty)\right) dt$$

$$= t\,f(tx,ty)\Big|_{t=0}^{t=1} = f(x,y)$$

Symmetry gives $\frac{\partial F}{\partial y} = g(x,y)$

Conclude: $dF = \omega$, as we wanted. $\square$

Remark: if ω is holomorphic (ie $\omega = f\,dz$ for $f \in \mathcal{O}(D_r(0))$) the proof of Proposition 1 is simpler! Write $f = \sum_{n=0}^{\infty} c_n z^n$ & set $F = \int \sum_{n=0}^{\infty} c_n z^n dz = \sum_{n=0}^{\infty} \frac{c_{n+1}}{n+1} z^{n+1}$

By construction $F \in \mathcal{O}(D_r(0))$ (Abel's Theorem) & $dF = \omega$.

- Primitives of closed differential forms only exist as multivalued functions (vary the constant!) This will lead to constructing new R.S from closed diff'l 1-forms: Prior, we define the sheaf of primitives of a closed form:

Definition: Given Y a RS & $\omega \in \mathcal{E}^{(1)}(Y)$ closed, we define a sheaf $\mathcal{F}$ as follows:

- For each $U \subseteq Y$ open, we set $\mathcal{F}(U) = \{ f \in \mathcal{E}(U) \mid df = \omega_{|U} \}$
- Restrictions are inherited from $\mathcal{E}$.

We call $\mathcal{F}$ the sheaf of primitives of ω

Lemma 3: $\mathcal{F}$ is a sheaf & it satisfies the Identity Theorem

PF/ Presheaf condition is clear. Since $\mathcal{F} \subseteq \mathcal{E}$ (sheaf) & the the differential condition is compatible with gluing in $\mathcal{E}$, we get that $\mathcal{F}$ is a sheaf.

Any $f_1, f_2 \in \mathcal{F}(U)$ for U open & connected must differ by a constant. Hence, if $\rho_y(f_1) = \rho_y(f_2)$ for some $y \in U$, we conclude the constant is 0 & so $f_1 = f_2$. $\square$

Theorem 1: Fix Y a R.S & $\omega \in \mathcal{E}^{(1)}(Y)$ closed. Then, there exist
① a RS X & a covering map $p: X \longrightarrow Y$
② a primitive $F \in \mathcal{E}(X)$ of the differential form $p^*\omega$.

Proof: Let $\mathcal{F}$ be the sheaf of primitives of ω & consider $p: |\mathcal{F}| \longrightarrow Y$, which we know is a surjective local homeomorphism (primitives exist locally by Proposition 1). Since Y is Hausdorff & locally connected (R.S) & $\mathcal{F}$ satisfies the Identity Thm, we know by Theorem 2 §8.2 that $|\mathcal{F}|$ is Hausdorff.

$\Rightarrow$ We use p local homeo to make $|\mathcal{F}|$ into a RS & p a holomorphic map

• **Claim 1:** p is a covering map.

Pf/ We check the definition. Pick $y \in Y$ & a local chart (U, φ) for y with $U \xrightarrow{\varphi} \mathbb{D}$, $y \mapsto 0$.
By Proposition 1, $\omega|_U$ has a primitive on U, call it $f \in \mathcal{E}(U)$. Furthermore $f+c$ is also a primitive for ω $\forall c \in \mathbb{C}$.

Thus, $$p^{-1}(U) = \bigsqcup_{c \in \mathbb{C}} \mathcal{N}(U, f+c) \subseteq |\mathcal{F}|$$ (The sets are pairwise disjoint by the Identity Thm)

& $p|_{\mathcal{N}(U,f+c)}: \mathcal{N}(U, f+c) \xrightarrow{\sim} U$ by construction (Theorem 2 §8.2)

• Any connected component X of $|\mathcal{F}|$ gives ①

• We define $F: X \longrightarrow \mathbb{C}$ by $F(x) = x(y) = x(p(x))$, $x \in \mathcal{F}_{Y,y}$

Claim 2: F is holomorphic & $dF = p^*\omega$

Pf/ Pick $x \in X$ & set $y = p(x) \in Y$. Fix $U \subseteq Y$ open connected with $y \in Y$, and a primitive f of $\omega|_U$ with $\rho_y(f) = x$. Since X is connected, $x \in \mathcal{N}(U, f) \subseteq X$

$$\begin{tikzcd} N(U,f) \arrow[d, "p"] \arrow[dr, "F"] & \\ \mathbb{D} \xleftarrow{\varphi} U \arrow[r, "f"'] & \mathbb{C} \end{tikzcd}$$

Since $F|_{\mathcal{N}(U,f)} = f\circ p|_{\mathcal{N}(U,f)}$ we see that F is locally holomorphic

$dF = d(f\circ p) = d(p^*f) = p^*(df) = p^*\omega$ by construction.

<u>Note</u>: The choice of X depends on a choice of a local primitive for ω near a point on Y.

⚠ $\mathcal{F}$ sheaf of primitives for ω need not have a global section. This is precisely what we are trying to build. (if $\mathcal{F}(Y)\neq\phi$, then $F\in\mathcal{F}(Y)$, $X=Y$ & $p=\mathrm{id}$ works)

<u>Example</u>: $Y=\mathbb{C}^*$ & $\omega=\frac{dz}{z}$ is a closed holomorphic 1-form on $\mathbb{C}^*$.
($d\omega = \frac{-1}{z^2}\,dz\wedge dz = 0$.)

The Triple $(X, p:X\to Y, F)$ given by the theorem agrees with the construction of $(X, p:X\to Y, F, b) = \Pi AC$ of $\log(z) = \sum_{r=1}^{\infty}\frac{(1-z)^r}{r}$ near 1; $p(b)=\log z$.
$F=\log: X\longrightarrow\mathbb{C}$ is in $\mathcal{O}(X)$ & it is a primitive for $p^*(\omega)$ on X

<u>Corollary 1</u>: Fix Y a RS & let $\pi:\tilde{Y}\longrightarrow Y$ be its universal cover. Assume $\omega\in\mathcal{E}^{(1)}(Y)$ is closed. Then $\exists\, G\in\mathcal{E}(\tilde{Y})$ with $dG=\pi^*\omega$.

<u>Proof</u>: Build $p: X\longrightarrow Y$ covering & $F\in\mathcal{E}(X)$ with $dF=p^*\omega$ using Theorem. By universal property $\exists\ \sigma: \tilde{Y}\xrightarrow{\sigma} X$ holo. Take $\boxed{G := F\circ\sigma = \sigma^*F}$ $\in \mathcal{E}(\tilde{Y})$

$$\begin{tikzcd} \tilde{Y} \arrow[rr, "\sigma"] \arrow[dr, "\pi"'] & \circlearrowleft & X \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

Then, $dG := d(\sigma^*F) = \sigma^*(dF) = \sigma^*p^*(\omega) = (p\circ\sigma)^*\omega = \pi^*\omega$ $\square$

<u>Corollary 2</u>: If Y is a simply connected RS, every closed diff'ble 1-form on X is exact.
(ie $\exists\, F\in\mathcal{E}(Y)$ with $dF=\omega$ $\Leftrightarrow$ $\omega\in\mathcal{E}^{(1)}(Y)$ is closed).
Pf/ $\mathrm{id}: Y\longrightarrow Y$ is the universal cover.

We can reinterpret path integrals $\int_\gamma \omega$ using this corollary.

Theorem 2: Fix X RS, $\pi : \tilde{X} \to X$ its universal cover. Fix $\omega \in \mathcal{E}^{(1)}(X)$ closed & $F \in \mathcal{E}(\tilde{X})$ primitive for $\pi^*\omega$. Given a piecewise diff'l path $\gamma : [0,1] \to X$ & a lifting $\hat{\gamma}$ of γ relative to π, we have $\int_\gamma \omega = F(\hat{\gamma}(1)) - F(\hat{\gamma}(0))$

Proof: Since coverings have the curve lifting property,

$$\begin{tikzcd} & \tilde{X} \arrow[d, "\pi \text{ covering}"] \\ {[0,1]} \arrow[ur, "\exists \hat{\gamma}"] \arrow[r, "\gamma"'] & X \end{tikzcd}$$

we can find $\hat{\gamma} : [0,1] \to \tilde{X}$ with $p \circ \hat{\gamma} = \gamma$

Now $\displaystyle\int_{\hat{\gamma}} \pi^* \omega = \int_{\pi \circ \hat{\gamma}} \omega$ by the definition of path integral

So $$\int_\gamma \omega = \int_{\pi \circ \hat{\gamma}} \omega = \int_{\hat{\gamma}} \pi^* \omega = \int_{\hat{\gamma}} dF \underset{\substack{\uparrow \\ \text{Thm §15.1}}}{=} F(\hat{\gamma}(1)) - F(\hat{\gamma}(0)).$$ □

Corollary 3: Fix X a R.S & $\omega \in \mathcal{E}^{(1)}(X)$ a closed diff'l 1-form. Then

(1) If $a, b \in X$ & $\gamma_1, \gamma_2 : [0,1] \to X$ a homotopic curves with $\gamma_1(0) = \gamma_1(0) = a$, $\gamma_2(1) = \gamma_2(1) = b$,

then $$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$$

(2) If $\gamma_1, \gamma_2 : [0,1] \to X$ are two closed curves that are free-homotopic

($H : [0,1] \times [0,1] \to X$ cont $H(0,t) = \gamma_1(t)$, $H(1,t) = \gamma_2(t)$, $H(s,0) = H(s,1)$ $\forall s$)

then $$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega.$$

Proof: (1) Consider $\pi : \tilde{X} \to X$ universal covering & liftings $\hat{\gamma}_1, \hat{\gamma}_2$ rel to π with the same initial point. Since coverings lift homotopies (Thm 1 §4.4 & Thm 2 §5.1) we see $\hat{\gamma}_1(1) = \hat{\gamma}_2(1)$ So by Theorem 2 $$\int_{\gamma_1} \omega = F(\hat{\gamma}_1(1)) - F(\hat{\gamma}_1(0))$$
where $F \in \mathcal{E}(\tilde{X})$ satisfies $dF = \pi^* \omega$ $$= F(\hat{\gamma}_2(1)) - F(\hat{\gamma}_2(0)) = \int_{\gamma_2} \omega.$$

(2) Set $a = \gamma_1(0) = \gamma_1(1)$ & $b = \gamma_2(0) = \gamma_2(1)$ & write $\alpha(s) = H(s,0) : [0,1] \to X$

This curve join a to b. By Lemma §4.4, $\gamma_1 \sim \alpha_0 * \gamma_2 * \alpha_0^-$

$$\Rightarrow \int_{\gamma_1} \omega \underset{\text{by (1)}}{=} \int_{\alpha_0 * \gamma_2 * \alpha_0^-} \omega = \int_{\alpha_0} \omega + \int_{\gamma_2} \omega + \int_{\alpha_0^-} \omega = \int_{\alpha_0} \omega + \int_{\gamma_2} \omega - \int_{\alpha_0} \omega = \int_{\gamma_2} \omega$$

$\square$

§15.3 Periods.

Definition: Fix X a RS & $\omega \in \mathcal{Z}^{(1)}(X)$ closed 1-form. For every $\gamma \in \pi_1(X, x_0)$ we define $a_\gamma := \int_\gamma \omega$

We call it a period of ω.

By Corollary 3 §15.2, this is well-defined (if $\gamma_1 \sim \gamma_2$, then $a_{\gamma_1} = a_{\gamma_2}$).

Also, it's independent of the choice of base pt (using $\gamma_1 = u * \gamma_2 * u^-$ (loops γ_1 at x_0, γ_2 at x_1, path u from x_0 to x_1) we have $\int_{\gamma_2} \omega = \int_{\gamma_1} \omega$).

As a consequence of the additivity of integrals, we have:

Proposition: We have a group homomorphism $\rho_\omega : \pi_1(X) \longrightarrow (\mathbb{C}, +)$, $\gamma \longmapsto \int_\gamma \omega$ for each closed 1-form $\omega \in \mathcal{Z}^{(1)}(X)$.

We call it the period homomorphism associated to the closed 1-form ω.

Example: $X = \mathbb{C}^*$ so $\pi_1(X) = \mathbb{Z}$. Pick $\gamma : [0,1] \to X$, $t \longmapsto e^{2\pi i t}$

We know $\pi_1(X) = \langle \gamma \rangle$

Fix $\omega = \frac{dz}{z}$ closed then $\int_\gamma \omega = \int_\gamma \frac{dz}{z} = 2\pi i$

The period homomorphism becomes $\rho_\omega(\gamma^n) = 2\pi i n$

Our next main result justifies the relevance of the period homomorphism.

Main Theorem: Fix X a RS & $\omega \in \mathcal{Z}^{(1)}(X)$ a closed form. Then:

$$\omega \text{ is exact} \iff \rho_\omega \equiv 0 : \pi_1(X) \longrightarrow (\mathbb{C}, +).$$

Idea: Pick $p: \tilde{X} \to X$ universal cover & find a primitive F to $p^*\omega$ on $\tilde{X}$ (this always exists by Corollary 1 §15.2). Next, we'll need to find necessary & sufficient conditions that ensure F descends to X. These will be precisely the vanishing of all periods of ω.

GOAL: Fix $F_0 \in \mathcal{E}(X)$ with $F_0 \circ p = F$ $\quad (p^* F_0 = F)$.

$$\begin{tikzcd} \tilde{X} \arrow[r, "F"] \arrow[d, "p"'] & \mathbb{C} \\ X \arrow[ur, "F_0"'] & \end{tikzcd}$$

We can do this $\iff$ F is constant along the fibers of p.
(ie F is fixed by $\mathrm{Deck}(\tilde{X}|X)$)

Proposition 1: Assume $f \in \mathcal{E}(\tilde{X})$ is fixed by $\mathrm{Deck}(\tilde{X}|X)$. Then $\exists\, f_0 \in \mathcal{E}(X)$ with $f = p^* f_0$. Furthermore, if f is holomorphic, so is f_0.

Proof: Given $x \in X$ & $\tilde{x} \in \pi^{-1}(x)$, pick $U \subseteq \tilde{X}$, $V \subseteq X$ opens with $\tilde{x} \in U$ & $x \in V$,
$p|_U : U \xrightarrow{\sim} V$ homeo (C^∞, by endowing $\tilde{X}$ with the manifold structure inherited from X via π)

Then: $\quad f_0|_V(x) = f(\tilde{x})$.

By construction, $f_0|_U := f \circ (p|_U)^{-1} = p|_U^* (f|_U)$ so it is in $C^\infty(U)$

Since f is invariant under $\mathrm{Deck}(\tilde{X}|X)$ & $p: \tilde{X} \to X$ is galois, f_0 is independent of the choice of $\tilde{x} \in p^{-1}(x)$. So f_0 is well-defined as a function on X.

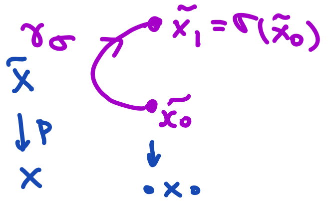
Since p is holomorphic, f_0 will be holomorphic whenever f is. $\square$

KEY p is a local homeomorphism, so $F_0 := p_* F$. $\iff$ $p^* F_0 = F$.

Proposition 2: Pick $\omega \in \mathcal{E}^{(1)}(X)$ closed & fix $F \in \mathcal{E}(\tilde{X})$ with $dF = p^*\omega$.
Then F is constant along the fibers of $p: \tilde{X} \to X$. if $\int \omega \equiv 0$

Proof: Given $x_0 \in X$ pick $\tilde{x}_0 = (x_0, [1_{x_0}])$ & $\tilde{x}_1 \in p^{-1}(x_0)$. Since p is a Galois covering $\exists\, \sigma \in \mathrm{Deck}(\tilde{X}|X)$ with $\tilde{x}_1 = \sigma(x_0)$.

$\tilde{X}$ is path connected so $\exists\, \gamma_\sigma : [0,1] \to \tilde{X}$ with $\gamma_\sigma(0) = \tilde{x}_0$, $\gamma_\sigma(1) = \tilde{x}_1$



This determines the group isomorphism

$$\Psi: \mathrm{Deck}(\tilde{X}|X) \xrightarrow{\simeq} \pi_1(X, x_0)$$
$$\sigma \longmapsto [p \circ \gamma_\sigma]$$

Then $0 = \rho_\omega(\gamma) = \int_\gamma \omega = \int_{\hat{\gamma}} p^*\omega = F(\hat{\gamma}_{(1)}) - F(\hat{\gamma}_{(0)}) = F(\tilde{x}_0) - F(\tilde{x}_1)$

by Theorem 2 §15.2

Conclusion: $F(\tilde{x}_0) = F(\tilde{x}_1)$ as we wanted.

Proof of Main Theorem: Since ω is closed, $\exists\, F \in \mathcal{E}(\tilde{X})$ primitive for $p^*\omega$ on $\tilde{X}$.

($\Rightarrow$) If ω is exact, then $\rho_\omega(\gamma) = \int_\gamma \omega = \int_\gamma dF = 0 \quad \forall \gamma \in \pi_1(X, x_0)$

($\Leftarrow$) Assume $\rho_\omega(\gamma) = 0 \quad \forall \gamma \in \pi_1(X, x_0)$. Then by Proposition 2, F is invariant under $\mathrm{Deck}(\tilde{X}/X)$. Proposition 1 builds $F_0 \in \mathcal{E}(X)$ with $F = p^* F_0$. Then $p^*(dF_0) = d(p^*F_0) = dF = p^*\omega$. Since p is a local homeomorphism, this forces $dF_0 = \omega$. (eg by working on local charts (U, z) of X)

Conclusion: ω is exact.

Corollary 1: If all the periods of ω vanish, we can build a special primitive for the closed form $\omega \in \mathcal{E}^{(1)}(X)$:

$$F(x) := \int_{x_0}^{x} \omega \qquad \text{where } x_0 \in X \text{ is fixed \& the integral is}$$

done along ANY path joining x_0 to x

Corollary 2: Fix a compact RS X & $\omega_1, \omega_2 \in \mathcal{E}^{(1)}(X)$ are holomorphic differential closed forms with $\rho_{\omega_1} = \rho_{\omega_2}$. Then $\omega_1 = \omega_2$.

Proof: $\omega := \omega_1 - \omega_2$ has vanishing periods, so ω has a primitive $F \in \mathcal{O}(X)$. But X is compact, so F is constant. Thus $\omega = dF = 0$. $\square$

§ 15.4 Summands of Automorphy

Fix X RS & $p : \tilde{X} \to X$ its universal cover

Recall $\operatorname{Deck}(\tilde{X}|X) \circlearrowright \mathcal{E}(\tilde{X})$ via $\sigma \cdot f = f \circ \sigma^{-1}$

This action is • linear $(\sigma \cdot (f+g) = \sigma \cdot f + \sigma \cdot g)$

• multiplicative $(\sigma \cdot (fg) = (\sigma \cdot f)(\sigma \cdot g))$

Definition: $f \in \mathcal{E}(\tilde{X})$ is called additively automorphic with constant summands of automorphy if $\exists\, a_\sigma \in \mathbb{C}$ s.t $\boxed{f - \sigma \cdot f = a_\sigma}$ $\forall \sigma \in \operatorname{Deck}(\tilde{X}|X)$

a_σ = summand of automorphy for σ

Example 1: If $f \in \mathcal{E}(\tilde{X})$ is fixed by $\operatorname{Deck}(\tilde{X}|X)$, then it's additively autom. $(a_\sigma = 0 \ \forall \sigma)$

By Prop 2 primitives of exact 1-forms are fixed by $\operatorname{Deck}(\tilde{X}|X)$

Example 2: $\omega = \frac{dz}{z}$ closed 1-form on $\mathbb{C}^*$, not exact on $\mathbb{C}^*$

$\log z = F : \widetilde{(\mathbb{C}^*)} \to \mathbb{C}$ is additively automorphic with constant summands of automorphy, and $a_\sigma = F - \sigma \cdot F = 2\pi i n$ if $\sigma \longleftrightarrow n$ under $\operatorname{Deck}(\tilde{\mathbb{C}}^* | \mathbb{C}^*) \simeq \pi_1(\mathbb{C}^*) = \mathbb{Z}$.

• Given $f \in \mathcal{E}(\tilde{X})$ & $\tilde{x}_0 \in \tilde{X}$, we define

$$A_f : \operatorname{Deck}(\tilde{X}|X) \longrightarrow (\mathbb{C}, +)$$
$$\sigma \longmapsto (f - \sigma \cdot f)(\tilde{x}_0)$$

Lemma: If f is additively automorphic, then A_f is a group homomorphism

Proof: Fix $\sigma, \tau \in \operatorname{Deck}(\tilde{X}|X)$, then

$$f - (\sigma \circ \tau^{-1}) \cdot f = \tau^{-1} \cdot f - \sigma \cdot (\tau^{-1} \cdot f) + f - \tau^{-1} \cdot f$$

$$\Rightarrow A_f(\sigma \circ \tau^{-1}) = A_{\tau^{-1} \cdot f}(\sigma) + A_f(\tau^{-1}) = A_f(\sigma) + A_f(\tau^{-1})$$

since $A_{\tau^{-1}\cdot f}(\sigma) = (\tau^{-1}\cdot f - \sigma\cdot\tau^{-1}\cdot f)(\tilde{x}_0) = (\underbrace{f - \sigma\cdot f}_{\text{constant}})(\tau(\tilde{x}_0)) = (f - \sigma\cdot f)(\tilde{x}_0) = A_f(\sigma)$ □

The next two results relate periods of closed forms to summands of automorphy.

Theorem Fix X a RS & $p: \tilde{X} \to X$ its universal cover. Fix $\omega \in \mathcal{E}^{(1)}(X)$ a closed form & $F \in \mathcal{E}(\tilde{X})$ with $dF = p^*\omega$. Then,

(1) F is additively automorphic with constant summands of automorphy.

(2) Under the isomorphism $\pi_1(X) \simeq \mathrm{Deck}(\tilde{X}|X)$, the summands of automorphy are the periods of ω.

Proof: (1) Pick any $\sigma \in \mathrm{Deck}(\tilde{X}|X)$. Then $p\circ\sigma^{-1} = p$ ensures $\sigma\cdot F$ is also a primitive for $p^*\omega$. Indeed:

$$\begin{tikzcd} \sigma \circlearrowright \tilde{X} \arrow[r, "F"] \arrow[d, "p"] & \mathbb{C} \\ X & \end{tikzcd}$$

$$d(\sigma\cdot F) = d(F\circ\sigma^{-1}) = d((\sigma^{-1})^*F) = (\sigma^{-1})^*(dF) = (\sigma^{-1})^*(p^*\omega) = (p\circ\sigma^{-1})^*\omega = p^*\omega.$$

Conclude: $a_\sigma := F - \sigma\cdot F$ is constant, so F is additively automorphic

(2) Need to check $a_\sigma = \int_{[p\circ\gamma_\sigma]} \omega$

But $\displaystyle\int_{[p\circ\gamma_\sigma]} \omega \underset{\text{Corollary 3 §15.3}}{=} \int_{p\circ\gamma_\sigma} \omega = F(\gamma_\sigma(1)) - F(\gamma_\sigma(0)) = F(\sigma(\tilde{x}_0)) - F(\tilde{x}_0)$

↳ γ_σ lifts $p\circ\gamma_\sigma$ rel to p Theorem 2 §15.2

$= -(\underbrace{F - \sigma^{-1}\cdot F}_{= a_{\sigma^{-1}}\ (\text{constant})})(\tilde{x}_0)$

$= -a_{\sigma^{-1}} \underset{A_F \text{ gp hom.}}{=} a_\sigma$ □

Theorem 2: Fix X a RS & $\pi: \tilde{X} \to X$ its universal cover. Fix $F \in \mathcal{E}(\tilde{X})$ an additively automorphic function with constant summands of automorphy. Then $\exists!\, \omega \in \mathcal{E}^{(1)}(X)$ closed with $dF = p^*\omega$.

Proof: Write $a_\sigma := F - \sigma \cdot F$. Then:

$$\sigma^*(dF) = d(\sigma^* F) = d(F \circ \sigma^{-1}) = d(F - a_\sigma) = dF$$

In particular dF is invariant under $\mathrm{Deck}(\tilde{X}|X)$

$$\begin{tikzcd} \mathcal{E}(X) \arrow[r, "d"] \arrow[d, "p^*"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{E}^{(1)}(X) \arrow[d, "p^*"] \\ \mathcal{E}(\tilde{X}) \arrow[r, "d"] & \mathcal{E}^{(1)}(\tilde{X}) \ni dF \end{tikzcd}$$

Now p is local homeo, so p^* is invertible & $(p^*)^{-1} = p_*$

Write $\omega := p_*(dF)$

- Locally on $a \in X$, pick (V, z) local chart $z: V \xrightarrow{\sim} D$, $a \mapsto 0$ & $\{U_i\}_{i \in I}$ opens on $\tilde{X}$ with $p^{-1}(V) \simeq \bigsqcup_{i \in I} U_i$ & $p|_{U_i}: U_i \xrightarrow{\sim} V$ homeo $\forall i$ (on U_i we can use the local chart $(U_i, z \circ p|_{U_i})$)

Since dF is invariant under $\mathrm{Deck}(\tilde{X}|X)$ $\quad dF|_{U_i} = d(F \circ \sigma^{-1})|_{\sigma(U_i)}$

So we set $\omega|_V = (p|_{U_i})_* (dF|_{U_i})$

By construction (RHS) is indep of the choice of $i \in I$ $\Rightarrow$ ω is well-defined

We can glue to $\omega \in \mathcal{E}^{(1)}(X)$ since $\mathcal{E}^{(1)}$ is a sheaf.

- Claim: $d\omega = 0$:

Pf/ $p^*(d\omega) = d p^*\omega = d dF = d^2 F = 0$ & p is a local homeomorphism.

- By construction $(p^*\omega)|_{U_i} = (p|_{U_i})^* \omega|_V = (p|_{U_i})^* (p|_{U_i})_* (dF|_{U_i}) = dF|_{U_i}$

Conclude: $p^*\omega = dF$

Uniqueness: any other ω' will have $p^*\omega = p^*\omega' = dF$ so $p^*(\omega - \omega') = 0$

Since p^* is local homeo, we get $\omega - \omega' = 0$.

Example: $\Gamma = \mathbb{Z}a + \mathbb{Z}b$ $\{a,b\}$ $\mathbb{R}$-linearly indep in $\mathbb{C}$ $\rightsquigarrow X = \mathbb{C}/\Gamma$ RS

$p: \mathbb{C} \longrightarrow \mathbb{C}/\Gamma$ is the universal covering & $\mathrm{Deck}(\mathbb{C}|X) \cong \Gamma$ (translations!)

$f = id: \mathbb{C} \longrightarrow \mathbb{C}$ is holo & additively automorphic with $a_\sigma = \sigma$ $\forall \sigma \in \Gamma$.

Thus, df is invariant under $\mathrm{Deck}(\mathbb{C}/X)$ & $\exists$ $\omega \in \mathcal{E}^{(1)}(X)$ holomorphic with $p^*\omega = dz$. Periods of ω are Γ. $\left(\int_{[p\circ\gamma_\sigma]} \omega = a_\sigma = \sigma\right)$

Q: Why study these automorphic functions with constant summands of automorphy?

A: Allow us to build holomorphic 1-forms on a non-compact Riemann surface with prescribed periods (Behnke-Stein). For this we work with

$p: X \longrightarrow Y$ holomorphic covering of Riemann surfaces (unbranched!)

& $\mathrm{Deck}(X|Y)$ [see §28 Forster]

$\rightsquigarrow a_\sigma = f - \sigma \circ f \in \mathcal{O}_{\mathcal{E}}(X)$ whenever $f \in \mathcal{O}_{\mathcal{E}}(X)$ & $\sigma \in \mathrm{Deck}(X|Y)$

- If p is Galois & $a_\sigma = 0$ $\forall \sigma$, then $f \in p^*\mathcal{O}_{\mathcal{E}}(X) \subseteq \mathcal{O}_{\mathcal{E}}(Y)$, which we identify with $\mathcal{O}_{\mathcal{E}}(X)$

Theorem (Behnke-Stein) If X is a non-compact RS & $\Phi: \pi_1(X) \longrightarrow \mathbb{C}$, $\gamma \longmapsto a_\gamma$ is a group homomorphism, then $\exists$ a holomorphic 1-form ω on X ($\Rightarrow$ closed) with $\int_\gamma \omega = a_\gamma$ $\forall \gamma \in \pi_1(X)$

Proof Build $F \in \mathcal{O}(\tilde{X})$ with summands of automorphy a_σ (this requires some work!) Then, dF is a holomorphic 1-form on $\tilde{X}$ with periods a_γ (use $\gamma = [p\circ\gamma_\sigma]$ for each $\sigma \in \mathrm{Deck}(\tilde{X}|X)$). The 1-form $\omega = p_*(dF)$ has the same periods.