Lecture XVI: Čech Cohomology of sheaves

Recall
- Defined $\int_\gamma \omega$ for γ = piecewise diff'ble path by charts + additivity
  $\omega \in \mathcal{E}^{(1)}(X)$ $X = RS$
- Defined closed & exact 1-forms $\omega \in \mathcal{E}^{(1)}(X)$
  ($d\omega = 0$) ($\exists f$ with $df = \omega$)
  f = primitive
  - closed $\not\Rightarrow$ exact
  - locally we can build primitives
- Constructed new Riemann surfaces $\subseteq |\mathcal{F}|$ ($\mathcal{F}$ = sheaf of local primitives for ω on X)
  & primitives $F: Y \longrightarrow \mathbb{C}$ ($Y \subseteq |\mathcal{F}| \xrightarrow{p} X$ covering)
- Build primitives on $\begin{array}{c}\tilde{X}\\ p\downarrow\\ X\end{array}$ (ie $F \in \mathcal{E}(\tilde{X})$ with $dF = p^*\omega$)

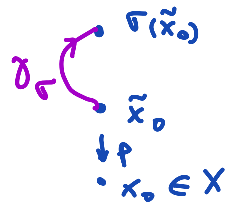
Consequences:
1. If ω is closed & X is simply connected $\Rightarrow$ ω is exact
2. $\int_\gamma \omega = F(\hat{\gamma}(1)) - F(\hat{\gamma}(0))$ for F prim for $p^*\omega$ on $\tilde{X}$
   $\hat{\gamma}$ lift of γ rel to p.
3. $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$ if $\gamma_1 \sim \gamma_2$ or γ_1 & γ_2 are free-homotopic loops.

$\rightsquigarrow$ Period homomorphism $\rho_\omega : \pi_1(X) \longrightarrow (\mathbb{C}, +)$
$\omega \in \mathcal{E}^{(1)}(X)$ closed $\gamma \longmapsto a_\gamma := \int_\gamma \omega$

Main Theorem: $\omega \in \mathcal{E}^{(1)}$ closed is exact $\Leftrightarrow \rho_\omega \equiv 0$

Corollary: If F is prim for $p^*\omega$ on $\tilde{X}$ & $\sigma \in \operatorname{Deck}(\tilde{X}|X)$, then

$$a_\sigma := F - \sigma_* F = F - F\circ\sigma^{-1} = \int_{[p\circ\gamma_\sigma]} \omega \quad \text{(constant!)}$$
$\in \mathcal{E}(\tilde{X})$



Application: Build holomorphic 1-forms on non-compact Riemann with prescribed periods.
(Behnke-Stein)

Next: Focus on cohomology theories with values on sheaves.

§16.1 Čech cohomology

We'll develop the theory for X = Top space & only specialize to X = R.S. when needed.

INPUT: X top space, $\mathcal{F}$ = sheaf on X (of ab gps / vector sp)

$\underline{\mathcal{U}} := \{U_i\}_{i\in I}$ open cover of X.

Next: we define cochains, cocycles & coboundaries.

Def: ① For $p\geq 0$: $C^p(\underline{\mathcal{U}},\mathcal{F}) = \prod_{\substack{i_0<i_1<\dots<i_p \\ \text{in } I}} \mathcal{F}(U_{i_0}\cap U_{i_1}\cap\dots\cap U_{i_p})$ (p^{th} cochain sp.)

② Coboundary maps:

$$\begin{array}{ccc} C^p(\underline{\mathcal{U}},\mathcal{F}) & \xrightarrow{\partial} & C^{p+1}(\underline{\mathcal{U}},\mathcal{F}) \\ \rotatebox{90}{$\in$} & & \\ (f_{\underline{i}})_{\underline{i}} = \tau & \longmapsto & \partial\tau = (g_{\underline{j}})_{\underline{j}} \end{array}$$

with $g_{i_0\dots i_{p+1}} := \sum_{k=0}^{p+1}(-1)^k \underbrace{f_{i_0,\dots,\hat{i}_k,\dots,i_{p+1}}}_{\in\mathcal{F}(U_{i_0}\cap\dots\cap\hat{U}_{i_k}\cap\dots\cap U_{i_{p+1}})}\Big|_{U_{i_0}\cap\dots\cap U_{i_{p+1}}} \in \mathcal{F}(U_{i_0}\cap\dots\cap U_{i_{p+1}})$

Write $\boxed{U_{i_0\dots i_p} = U_{i_0}\cap\dots\cap U_{i_p}}$

Example $p=0$: $\partial(f_i)_{i\in I} = \left(f_j|_{U_{ij}} - f_i|_{U_{ij}}\right)_{i<j}$

$p=1$ $\partial(f_{ij})_{i<j} = \left(f_{jk}|_{U_{ijk}} - f_{ik}|_{U_{ijk}} + f_{ij}|_{U_{ijk}}\right)_{i<j<k}$

Lemma 1: $\partial^2=0$ so we get a (co)chain complex $C^0(\underline{\mathcal{U}},\mathcal{F}) \xrightarrow{\partial} C^1(\underline{\mathcal{U}},\mathcal{F}) \xrightarrow{\partial} \dots$

Proof. The proof boils down to careful book keeping.

$$\underbrace{\partial^2(f_{\underline{i}})_{\underline{i}}}_{\cap\, C^p(\underline{\mathcal{U}},\mathcal{F})} = \partial(\partial f_{\underline{i}})\Big|_{U_{i_0,\dots,i_{p+3}}} = \sum_{k=0}^{p+3}(-1)^k\left(\partial(f_j)\right)_{i_0\dots\hat{i}_k\dots i_{p+3}}$$

$$= \sum_{k=0}^{p+3}(-1)^k\Big(\sum_{j=0}^{k-1}(-1)^j f_{i_0\dots\hat{i}_j\dots\hat{i}_k\dots i_{p+3}} + \sum_{j=k+1}^{p+3}(-1)^{j+1} f_{i_0\dots\hat{i}_k\dots\hat{i}_j\dots i_{p+3}}$$

i_k was skipped, so we shift by 1

$$= \sum_{k=0}^{p+3}\sum_{j=0}^{k-1}(-1)^{k+j} f_{i_0\dots\hat{i}_j\dots\hat{i}_k\dots i_{p+3}} - \sum_{k=0}^{p+3}\sum_{j=k+1}^{p+3}(-1)^{k+j} f_{i_0\dots\hat{i}_k\dots\hat{i}_j\dots i_{p+3}}$$

$$= \sum_{\substack{k,j=0\\ j<k}}^{p+3} (-1)^{k+j} f_{i_0 \ldots \hat{i}_j \ldots \hat{i}_k \ldots i_{p+3}} - \sum_{\substack{k,j=0\\ k<j}}^{p+3} (-1)^{k+j} f_{i_0 \ldots \hat{i}_k \ldots \hat{i}_j \ldots i_{p+3}}.$$

← swap k & j

$$= \sum_{\substack{k,j=0\\ j<k}}^{p+3} (-1)^{k+j} f_{i_0 \ldots \hat{i}_j \ldots \hat{i}_k \ldots i_{p+3}} - \sum_{\substack{k,j=0\\ j<k}}^{p+3} (-1)^{k+j} f_{i_0 \ldots \hat{i}_j \ldots \hat{i}_k \ldots i_{p+3}} = 0$$

Conclude: $\partial^2(\xi) = 0 \quad \forall \xi \in C^p(\underline{\mathcal{U}}, \mathcal{F})$ □

Def: We call $C^0(\underline{\mathcal{U}}, \mathcal{F}) \xrightarrow{\partial} C^1(\underline{\mathcal{U}}, \mathcal{F}) \xrightarrow{\partial} \cdots$ the Čech complex

Alt notation: Use $\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i<j} \mathcal{F}(U_{ij}) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \prod_{i<j<k} \mathcal{F}(U_{ijk}) \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow} \cdots$ &

Take alternating differences

We build the cohomology groups using cocycles & coboundaries

⇝ • $Z^p(\underline{\mathcal{U}}, \mathcal{F}) := \operatorname{Ker}\left(C^p(\underline{\mathcal{U}}, \mathcal{F}) \xrightarrow{\partial} C^{p+1}(\underline{\mathcal{U}}, \mathcal{F})\right)$ (cocycles)

• $B^{p+1}(\underline{\mathcal{U}}, \mathcal{F}) := \operatorname{Im}\left(C^p(\underline{\mathcal{U}}, \mathcal{F}) \xrightarrow{\partial} C^{p+1}(\underline{\mathcal{U}}, \mathcal{F})\right)$ (coboundaries)

We have $B^p(\underline{\mathcal{U}}, \mathcal{F}) \subseteq Z^p(\underline{\mathcal{U}}, \mathcal{F}) \quad \forall p \geq 1$ by Lemma 1, so we can build the Čech cohomology groups:

$$\boxed{H^0(\underline{\mathcal{U}}, \mathcal{F}) = Z^0(\underline{\mathcal{U}}, \mathcal{F}) \quad \& \quad H^p(\underline{\mathcal{U}}, \mathcal{F}) = Z^p(\underline{\mathcal{U}}, \mathcal{F}) / B^p(\underline{\mathcal{U}}, \mathcal{F}) \text{ for } p \geq 1}$$

Proposition 1: $H^0(\underline{\mathcal{U}}, \mathcal{F}) = \mathcal{F}(X)$ (cover independent!)

Proof: $Z^0(\underline{\mathcal{U}}, \mathcal{F}) = \{(f_i)_{i \in I} \mid f_i|_{U_{ij}} = f_j|_{U_{ij}}\}$ This is precisely the cocycle (gluing) condition for sheaves!

Examples: ① $Z^1(\underline{\mathcal{U}}, \mathcal{F}) = \{(f_{ij})_{i<j} : f_{jk}|_{U_{ijk}} - f_{ik}|_{U_{ijk}} + f_{ij}|_{U_{ijk}} \equiv 0 \ \forall i<j<k\}$

Equivalently: $f_{ik} = f_{ij} + f_{jk}$ on U_{ijk} (cocycle rel-n)

② $B^1(\underline{\mathcal{U}},\mathcal{F}) = \{ (f_{ij}) \ \exists (g_i)_i$ with $f_{ij} = g_{j|_{U_{ij}}} - g_{i|_{U_{ij}}} \}$

$= \{ (g_j - g_i)_{i<j} : g_k \in \mathcal{F}(U_k) \ \forall k \}$

$B^1(\underline{\mathcal{U}},\mathcal{F}) \subseteq Z^1(\underline{\mathcal{U}},\mathcal{F})$ since $(g_k - g_j) - (g_k - g_i) + (g_j - g_i) = 0$ on U_{ijk}

Observation: To free ourselves from the restriction of $i_0 < \cdots < i_p$ for indices of elements in $C^p(\underline{\mathcal{U}},\mathcal{F})$, we can use the cocycle relations to allow p-tuples with

- repetitions $\rightsquigarrow$ $f_{\underline{i}} = 0$ if $\underline{i}$ has repetitions
- unordered tuples $\rightsquigarrow$ $f_{\sigma(\underline{i})} = (-1)^{\mathrm{sign}(\sigma)} f_{\underline{i}}$

This simplifies the proofs, since we don't need to worry about ordering.

⚠ **Main issue:** the construction of $H^p(\underline{\mathcal{U}},\mathcal{F})$ for $p\geq 1$ is covering dependent!
To free ourselves from this & define $H^p(X,\mathcal{F})$ we need to see how Čech cohomology gps behave under refinement of open coverings.

§16.2 Refinements & Čech cohomology:

Definition: Given 2 open coverings $\underline{\mathcal{U}} = (U_i)_{i\in I}$, $\underline{\mathcal{V}} = (V_k)_{k\in K}$, we say $\underline{\mathcal{V}}$ is a refinement of $\underline{\mathcal{U}}$ if $\exists\ \tau: K \longrightarrow I$ st $V_k \subset U_{\tau(k)}$ $\forall k$.
$(\underline{\mathcal{V}} < \underline{\mathcal{U}})$

Consequence: If $\underline{\mathcal{V}}$ refines $\underline{\mathcal{U}}$, we can restrict sections further, since

$$V_{k_0\ldots k_p} = V_{k_0} \cap \cdots \cap V_{k_p} \subseteq U_{\tau k_0} \cap \cdots \cap U_{\tau k_p} := U_{\tau k_0 \cdots \tau k_p}$$

We get a map $\boxed{\tau^{\mathcal{U}}_{\mathcal{V}} : C^p(\underline{\mathcal{U}},\mathcal{F}) \longrightarrow C^p(\underline{\mathcal{V}},\mathcal{F})}$ (dependent of τ!)

$$(f_{\underline{i}}) \longmapsto (f_{\tau \underline{k}}|_{V_{\underline{k}}})_{\underline{k}}$$

Lemma 1: $\tau^{\mathcal{U}}_{\mathcal{V}}$ commutes with the coboundary maps ∂

Proof: • $\partial\Big(\tau^{\mathcal{U}}_{\mathcal{V}}\big(\underset{C^p(\underline{\mathcal{U}},\mathcal{F})}{(f_{\underline{i}})_{\underline{i}}}\big)\Big)\Big|_{k_0\cdots k_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \left(\tau^{\mathcal{U}}_{\mathcal{V}}(f_{\underline{i}})\right)_{k_0\cdots\hat{k}_j\cdots k_{p+1}}$

$$= \sum_{j=0}^{p+1} (-1)^j \left(f_{\tau k_0\cdots\tau\hat{k}_j\cdots\tau k_{p+1}}\right)\Big|_{V_{k_0\cdots\hat{k}_j\cdots k_{p+1}}} \quad \text{on } V_{k_0\cdots k_{p+1}}$$

• $\tau^{\mathcal{U}}_{\mathcal{V}}\left(\partial(f_{\underline{i}})_{\underline{i}}\right)\Big|_{k_0\cdots k_{p+1}} = \partial(f_{\underline{i}})_{\tau k_0\cdots\tau k_{p+1}}\Big|_{k_0\cdots k_{p+1}}$

$$= \sum_{j=0}^{p+1} (-1)^j \left(f_{\tau k_0\cdots\widehat{\tau k_j}\cdots\tau k_{p+1}}\right)\Big|_{k_0\cdots k_{p+1}} \quad \text{on } V_{k_0\cdots k_{p+1}}$$

So $\partial\,\tau^{\mathcal{U}}_{\mathcal{V}} = \tau^{\mathcal{U}}_{\underline{\mathcal{V}}}\,\partial$

Corollary 1: $\tau^{\mathcal{U}}_{\mathcal{V}}$ induces a map on cohomology $\tau^{\mathcal{U}}_{\mathcal{V}} : H^p(\underline{\mathcal{U}},\mathcal{F}) \longrightarrow H^p(\underline{\mathcal{V}},\mathcal{F})$

Proof: By Lemma 2, $\tau^{\mathcal{U}}_{\mathcal{V}}(Z^p(\underline{\mathcal{U}},\mathcal{F})) \subseteq Z^p(\underline{\mathcal{V}},\mathcal{F})$ $\quad \forall p \geq 0$.
& $\tau^{\mathcal{U}}_{\mathcal{V}}(B^p(\underline{\mathcal{U}},\mathcal{F})) \subseteq B^p(\underline{\mathcal{V}},\mathcal{F})$ $\quad \forall p \geq 1$ □

Q: Does this map depend on the choice of τ?

A: No!

Lemma 2: Assume $\underline{\mathcal{V}} < \underline{\mathcal{U}}$ via two maps $\tau, \tilde{\tau} : K \longrightarrow I$. Then, there is a homotopy operator between $\tau^{\mathcal{U}}_{\mathcal{V}}$ & $\tilde{\tau}^{\mathcal{U}}_{\mathcal{V}}$ at the cochain level.

Proof: We have

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & C^p(\underline{\mathcal{U}},\mathcal{F}) \arrow[r, "\partial"] \arrow[d, shift right, "\tau^{\mathcal{U}}_{\mathcal{V}}"'] \arrow[d, shift left, "\tilde{\tau}^{\mathcal{U}}_{\mathcal{V}}"] \arrow[dl, dotted] & C^{p+1}(\underline{\mathcal{U}},\mathcal{F}) \arrow[r, "\partial"] \arrow[d, shift right, "\tau^{\mathcal{U}}_{\mathcal{V}}"'] \arrow[d, shift left, "\tilde{\tau}^{\mathcal{U}}_{\mathcal{V}}"] \arrow[dl, "\Psi"'] & C^{p+2}(\underline{\mathcal{U}},\mathcal{F}) \arrow[r, "\partial"] \arrow[d, shift right, "\tau^{\mathcal{U}}_{\mathcal{V}}"'] \arrow[d, shift left, "\tilde{\tau}^{\mathcal{U}}_{\mathcal{V}}"] \arrow[dl, "\Psi"'] & \cdots \arrow[dl, dotted] \\
\cdots \arrow[r, "\partial"] & C^p(\underline{\mathcal{V}},\mathcal{F}) \arrow[r, "\partial"] & C^{p+1}(\underline{\mathcal{V}},\mathcal{F}) \arrow[r, "\partial"] & C^{p+1}(\underline{\mathcal{V}},\mathcal{F}) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

Define $\Psi : C^{p+1}(\underline{\mathcal{U}},\mathcal{F}) \longrightarrow C^p(\underline{\mathcal{V}},\mathcal{F})$ $\quad \forall p \geq 0$ $\quad$ via

$$\Psi(\xi)\big|_{V_{k_0\cdots k_p}} = \sum_{j=0}^{p} (-1)^j \, \xi_{\tau k_0 \cdots \tau k_j \tilde\tau k_j \tilde\tau k_{j+1} \cdots \tilde\tau k_p}\big|_{V_{k_0\cdots k_p}}$$

repeated index!

Claim: $\tilde\tau^{\mathcal U}_{\mathcal V} - \tau^{\mathcal U}_{\mathcal V} = \partial\Psi + \Psi\partial$

Pf/ By direct calculation. We evaluate on $C^{p+1}(\underline{\mathcal U},\mathcal F)$ so both sides lie in $C^{p+1}(\underline{\mathcal V},\mathcal F)$

$$(\tilde\tau^{\mathcal U}_{\mathcal V} - \tilde\tau^{\mathcal U}_{\mathcal V})(\underline f)\big|_{V_{k_0\cdots k_{p+1}}} = f\big|_{U_{\tilde\tau k_0\cdots \tilde\tau k_{p+1}}} - f\big|_{U_{\tau k_0\cdots \tau k_{p+1}}} \quad \text{on } V_{k_0\cdots k_{p+1}}$$

$$(\partial\Psi)(f)\big|_{V_{k_0\cdots k_{p+1}}} = \sum_{j=0}^{p+1} (-1)^j \, \Psi(f)\big|_{V_{k_0\cdots \hat k_j \cdots k_{p+1}}} = ① + ② \quad \text{where}$$

$$① = \sum_{j=0}^{p+1} \sum_{s=0}^{j-1} (-1)^{s+j} f_{\tau k_0 \cdots \tau k_s \tilde\tau k_s \cdots \hat{\tilde\tau} k_j \cdots \tilde\tau k_{p+1}} \quad \text{on } V_{k_0\cdots k_{p+1}}$$

$$② = \sum_{j=0}^{p+1} \sum_{s=j+1}^{p} (-1)^{j+s+1} f_{\tau k_0 \cdots \hat{\tau k_j} \cdots \tau k_s \tilde\tau k_{s+1} \cdots \tilde\tau k_{p+1}} \quad \text{on } V_{k_0\cdots k_{p+1}}$$

$$(\Psi\partial)(f)\big|_{V_{k_0\cdots V_{p+1}}} = \sum_{s=0}^{p+1} (-1)^s (\partial f)_{\tau k_0 \cdots \tau k_s \tilde\tau k_s \cdots \tilde\tau k_{p+1}} = ③ + ④ + ⑤ \quad \text{where}$$

$$③ = \sum_{s=0}^{p+1} \sum_{j=0}^{s-1} (-1)^{s+j} f_{\tau k_0 \cdots \hat{\tau k_j} \cdots \tau k_s \tilde\tau k_s \cdots \tilde\tau k_{p+1}}$$

$$④ = \sum_{s=0}^{p+1} \sum_{j=s+1}^{p} (-1)^{j+s+1} f_{\tau k_0 \cdots \tau k_s \tilde\tau k_s \cdots \hat{\tilde\tau} k_j \cdots \tau k_{p+1}}$$

$$⑤ = \sum_{s=0}^{p+1} \underset{1}{\cancel{(-1)^{s+s}}} f_{\tau k_0 \cdots \tau k_{s-1} \hat{\tau k_s} \tilde\tau k_s \cdots \tau k_{p+1}} + \underset{-1}{\cancel{(-1)^{s+(s+1)}}} f_{\tau k_0 \cdots \tau k_s \hat{\tilde\tau k_s} \tilde\tau k_{s+1} \cdots \tilde\tau k_{p+1}}$$

By construction ① $= -$ ④ & ③ $= -$ ②, so $(\partial\Psi + \Psi\partial) f\big|_{V_{k_0\cdots k_{p+1}}} =$ ⑤

By ⑤ is a telescopic sum, so we get $f_{\tilde\tau k_0 \cdots \tilde\tau k_{p+1}} - f_{\tau k_0 \cdots \tau k_{p+1}}$, as we wanted.

Corollary 2: $\tau^{\mathcal U}_{\mathcal V}$ & $\tilde\tau^{\mathcal U}_{\mathcal V}$ induce the same map on homology $H^p(\underline{\mathcal U},\mathcal F) \longrightarrow H^p(\underline{\mathcal V},\mathcal F)$

Lemma 3: Given 3 open coverings $\underline{\mathcal U}, \underline{\mathcal V}, \underline{\mathcal W}$ with $\underline{\mathcal W} < \underline{\mathcal V} < \underline{\mathcal U}$, we have

$$\tau^{\mathcal V}_{\mathcal W} \circ \tau^{\mathcal U}_{\mathcal V} = \tau^{\mathcal U}_{\mathcal W} \quad \text{on cohomology}$$

Proof We write $\mathcal{W} = \{W_j\}_{j\in J}$, $\mathcal{V} = \{V_k\}_{k\in K}$ & $\underline{\mathcal{U}} = \{U_i\}_{i\in I}$, & take refinements $\tau : J \longrightarrow K$, $\tau' : K \longrightarrow I$. Then $\tilde{\tau} := \tau' \circ \tau : J \longrightarrow I$ is a refinement & on cochains we get $\tau^{\mathcal{V}}_{\mathcal{W}} \circ \tau^{\mathcal{U}}_{\mathcal{V}} = \tilde{\tau}^{\mathcal{U}}_{\mathcal{W}}$. Indeed,

$$\tau^{\mathcal{V}}_{\mathcal{W}} \circ \tau^{\mathcal{U}}_{\mathcal{V}} (\underline{f})_{j_0 \cdots j_p} = (\tau^{\mathcal{U}}_{\mathcal{V}} f)_{\tau j_0 \cdots \tau j_p} \Big|_{W_{j_0 \cdots j_p}} = \left(f_{(\tau'\circ\tau)_{j_0} \cdots (\tau'\circ\tau)_{j_p}} \Big|_{V_{\tau j_0 \cdots \tau j_p}} \right) \Big|_{W_{0 \cdots j_p}}$$

$$= f_{\tilde{\tau} j_0 \cdots \tilde{\tau} j_p} \Big|_{W_{0\cdots j_p}}$$

by the restriction axiom on pre-sheaves.

The result follows since the map on cohomology is independent of the choice of refinement. $\square$

Theorem: $H^1(\underline{\mathcal{U}}, \mathcal{F}) \longrightarrow H^1(\underline{\mathcal{V}}, \mathcal{F})$ is injective (false for $p>1$)

Proof: Pick a map $\tau : K \longrightarrow I$ inducing $\underline{\mathcal{U}} < \underline{\mathcal{V}}$ & $f \in Z^1(\underline{\mathcal{U}}, \mathcal{F})$

Want to show that if $\bar{f} \in H^1(\underline{\mathcal{U}}, \mathcal{F})$ maps to 0 under $\bar{\tau}^{\mathcal{U}}_{\mathcal{V}}$, then $f = 0$

Equivalently, if $\tau^{\mathcal{U}}_{\mathcal{V}}(f) \in B^1(\underline{\mathcal{V}}, \mathcal{F})$, then $f \in B^1(\mathcal{U}, \mathcal{F})$

Pick $g \in C^0(\underline{\mathcal{V}}, \mathcal{F})$ with $\tau^{\mathcal{U}}_{\mathcal{V}}(f) = \partial g$

Want to find $\tilde{g} \in C^0(\underline{\mathcal{U}}, \mathcal{F})$ with $f = \partial \tilde{g}$. We need to determine $\tilde{g}$ on U_i $\underline{\forall}\, i$

- $\tau^{\mathcal{U}}_{\mathcal{V}}(f)\Big|_{V_{k_1,k_2}} = f_{\tau k_1 \tau k_2}$ on V_{k_1,k_2} & $\partial g\Big|_{V_{k_1,k_2}} = g_{k_2} - g_{k_1}$ on V_{k_1,k_2}

In order to determine $\tilde{g}_i \in \mathcal{F}(U_i)$ we cover U_i with $(U_i \cap V_k)_{k\in K}$

We define $h_{ik} := -f_{i\tau k} + g_k$ on $U_i \cap V_k$.

Claim 1: $h_{ik_1}\big|_{U_i \cap V_{k_1} \cap V_{k_2}} = h_{ik_2}\big|_{U_i \cap V_{k_1} \cap V_{k_2}}$ $\forall\, k_1, k_2$

Pf/ To show: $-f_{i\tau_{k_1}} + g_{k_1} \overset{?}{=} -f_{i\tau_{k_2}} + g_{k_2}$

$-(f_{i\tau_{k_1}} - f_{i\tau_{k_2}}) \overset{?}{=} g_{k_2} - g_{k_1}$ (*)

But $f \in Z^1(\mathcal{U}, \mathcal{F})$ so $\partial f|_{\mathcal{U}_{i\tau_{k_1}\tau_{k_2}}} = f_{\tau_{k_1}\tau_{k_2}} - f_{i\tau_{k_2}} + f_{i\tau_{k_1}} = 0$ on $\mathcal{U}_{i\tau_{k_1}\tau_{k_2}} \supseteq U_i \cap V_{k_1} \cap V_{k_2}$

Thus, (*) become $f_{\tau_{k_1}\tau_{k_2}} = g_{k_2} - g_{k_1}$ which does hold since $\tau^{\mathcal{U}}_{\mathcal{V}}(f) = \partial g$ $\square$

As a consequence, the sections $h_{ik} \in \mathcal{F}(U_i \cap V_k)$ glue together to $\tilde{g}_i \in \mathcal{F}(U_i)$ (sheaf axiom!)

- Claim 2: $\partial \tilde{g} = f$

Pf/ To show: $\partial(\tilde{g})|_{U_{i_0 i_1}} = \tilde{g}_{i_1} - \tilde{g}_{i_0} = f_{i_0 i_1}$ on $U_{i_0 i_1}$.

It's enough to check this on $U_{i_0 i_1} \cap V_k$ $\forall k \in K$. (conclude by sheaf axiom)

Note: $f_{i_0 i_1} = f_{i_0 \tau_k} + f_{\tau_k i_1}$ since $(\partial f)_{i_0 \tau_k i_1} = 0$

$\Rightarrow f_{i_0 i_1} = f_{i_0\tau_k} + f_{\tau_k i_1} = f_{i_0 \tau_k} - g_k + g_k + f_{\tau_k i_1} = \underbrace{f_{i_0\tau_k} - g_k}_{= -h_{i_0 k}} + \underbrace{g_k - f_{i_1 \tau_k}}_{+h_{i_1 k}}$

$= \tilde{g}_{i_1} - \tilde{g}_{i_0}$ on $U_{i_0 i_1} \cap V_k$ as desired. $\square$

§16.3 The definition of $H^1(X, \mathcal{F})$:

Using the refinement maps $H^1(\underline{\mathcal{U}}, \mathcal{F}) \xrightarrow{\tau^{\mathcal{U}}_{\mathcal{V}}} H^1(\underline{\mathcal{V}}, \mathcal{F})$ whenever $\underline{\mathcal{V}} < \underline{\mathcal{U}}$, we define an equivalence relation on $\bigsqcup_{\underline{\mathcal{U}}} H^1(\underline{\mathcal{U}}, \mathcal{F})$:

Def.: ξ in $H^1(\underline{\mathcal{U}}, \mathcal{F})$, η in $H^1(\underline{\mathcal{V}}, \mathcal{F})$ are equivalent (write $\xi \sim \eta$) if, and only if

$\exists\, \underline{\mathcal{W}}$ open covering with $\underline{\mathcal{W}} < \underline{\mathcal{V}}$ & $\underline{\mathcal{W}} < \mathcal{U}$ with $\tau^{\mathcal{U}}_{\mathcal{W}}(\xi) = \tau^{\mathcal{V}}_{\mathcal{W}}(\eta)$

• **Lemma1:** $\sim$ defines an equivalence relation on $\bigsqcup_{\underline{\mathcal{U}}} H^1(\underline{\mathcal{U}}, \mathcal{F})$

Proof For transitivity, we use $\tau^{\mathcal{V}}_{\mathcal{W}} \circ \tau^{\mathcal{U}}_{\mathcal{V}} = \tau^{\mathcal{U}}_{\mathcal{W}}$ (see Lemma 3 §16.2)

• As a consequence, we have an inductive (direct) limit definition for $H^1(X, \mathcal{F})$

Definition: $H^1(X,\mathcal{F}) = \varinjlim_{\underline{\mathcal{U}}} H^1(\underline{\mathcal{U}},\mathcal{F}) = \left(\bigsqcup_{\underline{\mathcal{U}}} H^1(\underline{\mathcal{U}},\mathcal{F}) \big/ \sim \right)$

Note: The same method allows to define $H^p(X,\mathcal{F}) = \varinjlim_{\underline{\mathcal{U}}} H^p(\underline{\mathcal{U}},\mathcal{F}) = \bigsqcup_{\underline{\mathcal{U}}} H^p(\underline{\mathcal{U}},\mathcal{F})\big/\sim$
(with the same type of equivalence relation.

Proposition 1: $H^1(X,\mathcal{F})$ an ab group / vector space if $\mathcal{F}$ is a sheaf of ab grps / vector spaces

Proof: Given $\xi \in H^1(\underline{\mathcal{U}},\mathcal{F})$ & $\eta \in H^1(\underline{\mathcal{V}},\mathcal{F})$, take $\underline{\mathcal{W}}$ a common refinement of $\underline{\mathcal{U}}$ & $\underline{\mathcal{V}}$ (Eg $\underline{\mathcal{W}} = \underline{\mathcal{U}} \sqcup \underline{\mathcal{V}}$ or $\mathcal{W} = \{ U_i \cap V_j \;\; \begin{smallmatrix} i\in I \\ j\in J\end{smallmatrix} \}$) & $\begin{matrix} \tau: K \longrightarrow I \\ \tilde{\tau}: K \longrightarrow J \end{matrix}$

Then $\xi + \eta$ = equiv class of $\tau^{\mathcal{U}}_{\mathcal{W}}(\xi) + \tilde{\tau}^{\mathcal{V}}_{\mathcal{W}}(\eta)$

Check: Independence of choices

• Neutral element $\underline{0} \in H^1(\underline{\mathcal{U}},\mathcal{F})$ for some $\underline{\mathcal{U}}$. □

Proposition 2: $H^1(\underline{\mathcal{U}},\mathcal{F}) \longrightarrow H^1(X,\mathcal{F})$ is injective for every open covering $\underline{\mathcal{U}}$.

Proof: By Theorem §16.2 $H^1(\underline{\mathcal{U}},\mathcal{F}) \xrightarrow{\tau^{\mathcal{U}}_{\mathcal{V}}} H^1(\underline{\mathcal{V}},\mathcal{F})$ is injective $\forall\, \underline{\mathcal{V}} < \underline{\mathcal{U}}$.

Thus $\xi \in H^1(\mathcal{U},\mathcal{F})$ is 0 in $H^1(X,\mathcal{F}) \Leftrightarrow \tau^{\mathcal{U}}_{\mathcal{V}}(\xi) = 0$ for $\underline{\mathcal{V}} \ll \underline{\mathcal{U}} \overset{\text{injectivity}}{\Leftrightarrow} \xi = 0$. □

Corollary: $H^1(X,\mathcal{F}) = 0 \Leftrightarrow H^1(\underline{\mathcal{U}},\mathcal{F}) = 0 \;\; \forall\, \underline{\mathcal{U}}$ open covering on X

§16.4 Cohomology of Riemann surfaces:

• Next, we fix X to be a Riemann surface & compute $H^1(X,\mathcal{F})$ for various sheaves

Main Theorem: If X is a compact Riemann surface, then $\dim_{\mathbb{C}} H^1(X,\mathcal{O})$ is finite

• We'll take this statement in faith (see §13 & 14 in Forster's book)

<u>Definition</u>: If X is a compact Riemann surface $g := \dim_{\mathbb{C}} H^1(X, \mathcal{O}) =$ genus of X

• To compute various $H^1(X, \mathcal{F})$ it's useful to work with partitions of unity

<u>Def</u> Given X diff'ble manifold & $\mathcal{U} = \{U_i\}_{i\in I}$ open covering, a differentiable partition of unity subordinate to $\mathcal{U}$ is a family $\{f_i\}_{i\in I}$ of differentiable functions $f_i : X \to \mathbb{R}$ satisfying the following 4 properties:

(1) $0 \leq f_i \leq 1 \quad \forall\, i \in I$

(2) $\operatorname{Supp} f_i \subseteq U_i \quad \forall\, i \in I$

(3) $\{\operatorname{supp} f_i\}_{i\in I}$ is locally finite, ie $\forall\, a \in X\ \exists\ V$ open with $a \in V$ st.

$V \cap \operatorname{Supp} f_i \neq \emptyset$ for only finitely many indices $i \in I$.

(4) $\sum_{i\in I} f_i \equiv 1$ (evaluating at each $x \in X$ is a finite sum!)

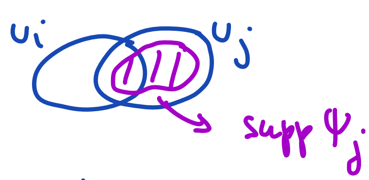
<u>Theorem 1</u>: If the topology of X is second countable (countable basis for the topology) (ok for R.S. by a Thm of Rado), any open covering $\mathcal{U}$ admits a subordinate partition of unity subordinate to it.

<u>Theorem 2</u>: For any R.S. X we have $H^1(\mathcal{U}, \mathcal{E}) = 0$ for all open coverings $\mathcal{U}$

<u>Proof</u>: A partition of unity (ψ_i) subordinate to $(U_i)_{i\in I}$.

Pick $f \in Z^1(\mathcal{U}, \mathcal{E})$, so $\partial f|_{i_0 i_1 i_2} = f_{i_1 i_2} - f_{i_0 i_2} + f_{i_0 i_1} = 0$ on $U_{i_0 i_1}$

Set $g_i = -\sum_{j\in I} \psi_j|_{U_i} \underbrace{f_{ij}}_{\text{defined on } U_{ij}} \quad \forall i$



• <u>Note</u> For each $x \in U_i$, the sum is finite in a neighborhood V of x.

In addition $\psi_j|_{U_i} f_{ij}|_V = \psi_j|_V f_{ij} \in \mathcal{E}(V)$ ($=0$ outside U_{ij})

So $g_i \in \mathcal{E}(U_i)$ $\forall i$.

<u>Claim</u>: $\partial g = f$ so $f \in B^1(\underline{\mathcal{U}}, \mathcal{E})$

Pf/ $g_{i_1} - g_{i_0} = -\sum_{j \in J} \varphi_j f_{i_1 j} + \sum_j \varphi_j f_{i_0 j} = -\sum_{j\in J} \varphi_j (f_{i_1 j} - f_{i_0 j})$

$= -\sum_{j \in J} \varphi_j \underbrace{f_{i_1 i_0}}_{\text{on } U_{i_0 i_1 j}} = -\left(\sum_{j \in J} \varphi_j\right)(-f_{i_0 i_1}) = f_{i_0 i_1}$ on $U_{i_0 i_1}$

<u>Conclude</u> $f = 0$ in $H^1(\underline{\mathcal{U}}, \mathcal{E})$ □

Same proof gives:

<u>Theorem 3</u>: For any R.S X & any open covering $\underline{\mathcal{U}}$ of X we have.

$$H^1(\underline{\mathcal{U}}, \mathcal{E}^{(1,0)}) = H^1(\underline{\mathcal{U}}, \mathcal{E}^{(0,1)}) = H^1(\underline{\mathcal{U}}, \mathcal{E}^{(1)}) = H^1(\underline{\mathcal{U}}, \mathcal{E}^{(2)}) = 0$$

The sheaves are modules over $\mathcal{E}$ (use partition of unity to alter the corresponding form, perhaps refining to cover by open charts to simplify notation)

<u>Corollary</u>: For any Riemann surface we have

$$H^1(X, \mathcal{E}) = H^1(X, \mathcal{E}^{(1)}) = H^1(X, \mathcal{E}^{(2)}) = H^1(X, \mathcal{E}^{(1,0)}) = H^1(X, \mathcal{E}^{(0,1)}) = 0$$

<u>Proof</u>: Use Theorems 2 & 3 & Corollary §16.3.