Lecture XVII: Čech Cohomology of sheaves II

Recall: X top space, $\underline{\mathcal{U}} = (U_i)_{i\in I}$ open cover, $\mathcal{F}$ = sheaf of ab gps / vector sp on X

- **p-cochains:** $C^p(\underline{\mathcal{U}}, \mathcal{F}) = \prod_{\substack{i_0<\dots<i_p \\ \text{in } I}} \mathcal{F}(U_{i_0\dots i_p})$ $\qquad U_{i_0,\dots,i_p} = U_{i_0}\cap\dots\cap U_{i_p}$

- **coboundary map:** $\partial: C^p(\underline{\mathcal{U}},\mathcal{F}) \longrightarrow C^{p+1}(\underline{\mathcal{U}},\mathcal{F})$ $\quad (\partial f)_{i_0\dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j f_{i_0\dots\hat{i_j}\dots i_{p+1}}\Big|_{U_{i_0\dots i_{p+1}}}$

- $\partial^2 = 0 \Rightarrow B^p(\underline{\mathcal{U}},\mathcal{F}) = \operatorname{im}\left(\partial|_{C^{p-1}(\underline{\mathcal{U}},\mathcal{F})}\right) \subseteq \underbrace{Z^p(\underline{\mathcal{U}},\mathcal{F})}_{\text{cocycles}} = \ker \partial|_{C^p(\underline{\mathcal{U}},\mathcal{F})}$

Convention $f_{\underline{i}} = 0$ if $\underline{i}$ has repeated indices ; $f_{\sigma(\underline{i})} = (-1)^{\operatorname{sign}(\sigma)} f_{\underline{i}}$ $\forall \sigma$ perm. of $\underline{i}$.

$$H^p(\underline{\mathcal{U}},\mathcal{F}) = Z^p(\underline{\mathcal{U}},\mathcal{F}) \big/ B^p(\underline{\mathcal{U}},\mathcal{F}) \quad \text{is ab gp / vector space}$$

- If $\underline{\mathcal{V}} < \underline{\mathcal{U}}$ refinement, we get $H^p(\underline{\mathcal{U}},\mathcal{F}) \xrightarrow{\tau^{\mathcal{U}}_{\mathcal{V}}} H^p(\underline{\mathcal{V}},\mathcal{F})$

  $\left(\begin{matrix}\underline{\mathcal{V}} = (V_k)_{k\in K} \\ \underline{\mathcal{U}} = (U_i)_{i\in I}\end{matrix} \quad \tau: K \to I \quad V_k \subseteq U_{\tau k}\right)$ $\qquad [\underline{f}] \longmapsto [\tau^{\mathcal{U}}_{\mathcal{V}}(f)] = [(f_{\tau k_0 \dots \tau k_p})_{k_0\dots k_p}]$

Key: This map is injective for $p=1$ & independent of τ.

- $H^1(X,\mathcal{F}) := \varinjlim_{\mathcal{U}} H^1(\underline{\mathcal{U}},\mathcal{F})$ & $H^1(X,\mathcal{F}) = 0 \iff H^1(\underline{\mathcal{U}},\mathcal{F}) = 0$ $\forall \underline{\mathcal{U}}$ open covering

Theorem: X R.S then $H^1(\underline{\mathcal{U}}, \mathcal{E}^{(k)}) = 0$ for all $k = 0, 1, (1,0), (0,1), 2$ $\forall \underline{\mathcal{U}}$ open covering

$H^1(X, \mathcal{E}^{(k)}) = 0$

§17.1 More cohomology examples:

Theorem 1: Assume X is a simply connected R.S. & fix $\underline{\mathcal{U}}$ open covering Then

$$H^1(\underline{\mathcal{U}}, \underline{\mathbb{C}}) = H^1(\underline{\mathcal{U}}, \underline{\mathbb{Z}}) = 0,$$

where $\underline{\mathbb{C}}$ & $\underline{\mathbb{Z}}$ are sheaves of locally constant functions with values in $\mathbb{C}$, resp $\mathbb{Z}$.

In particular, $H^1(X,\underline{\mathbb{C}}) = H^1(X,\underline{\mathbb{Z}}) = 0$

Proof. (1) First we argue for $\mathcal{F} = \underline{\mathbb{C}} \subseteq \mathcal{E}$.

Given $(c)_{ij} \in Z^1(\underline{\mathcal{U}}, \underline{\mathbb{C}})$, we can find $g \in C^0(\underline{\mathcal{U}}, \mathcal{E})$ with $\partial g = c$, ie

$g_j - g_i = c_{ij}$ is constant on U_{ij}.

Thus, $dg_j = dg_i$ on U_{ij} meaning we can find a global 1-form ω on X with $\omega_{|U_i} = dg_i$. In particular $d\omega = 0$ (ω is closed)

But X is simply connected, so ω is exact. Pick $f \in \mathcal{E}(X)$ with $df = \omega$.

Set $c_i := -f_{|U_i} + g_i$ on U_i

- $dc_i = -df_{|U_i} + dg_i = -\omega_{|U_i} + \omega_{|U_i} = 0$ so c_i is locally constant on U_i, that is $c_i \in \underline{\mathbb{C}}(U_i)$.

Now $c_j - c_i = g_j - g_i = c_{ij}$ on U_{ij} so $\partial(c) = (c_{ij})$ as we wanted

(2) For $\mathcal{F} = \underline{\mathbb{Z}}$, we need to use $e^{2\pi i z}$.

Given $(a_{jk}) \in Z^1(\underline{\mathcal{U}}, \underline{\mathbb{Z}}) \subseteq Z^1(\underline{\mathcal{U}}, \underline{\mathbb{C}})$ so we can find $(c_i)_i \in C^0(\underline{\mathcal{U}}, \underline{\mathbb{C}})$ with $a_{jk} = c_k - c_j$ on $U_j \cap U_k$

Take $\exp(2\pi i z)$ to get $1 = \exp(2\pi i a_{jk}) = \exp(2\pi i c_k)/\exp(2\pi i c_j)$

Conclusion: $\exp(2\pi i c_k) = \exp(2\pi i c_j)$ on $U_j \cap U_k$

Thus, there exists a locally constant function $f \in \underline{\mathbb{C}}(X)$ with $f_{|U_j} = \exp(2\pi i c_j)$

But X is connected, so this section is in fact a constant function

Choose $c \in \mathbb{C}$ with $\exp(2\pi i c) = f$ & set $a_j = -c_{|U_j} + c_j$ on U_j

This is a locally constant function on U_j, ie $a_j \in \underline{\mathbb{C}}(U_j)$.

- $\exp(2\pi i a_j) = \exp(2\pi i c_j)/\exp(2\pi i c) = f/f = 1$ on U_j so

$\Rightarrow a_j \in \underline{\mathbb{Z}}(U_j)$

• $a_k - a_j = c_k - c_j = a_{jk}$ on $U_j \cap U_k$. ie $\partial(a_j) = a_{jk}$.

Conclusion: $(a_{jk}) \in B^1(\underline{\mathcal{U}}, \underline{\mathbb{Z}})$, so $(a_{jk}) = 0$ in $H^1(\underline{\mathcal{U}}, \underline{\mathbb{Z}})$. $\square$

§17.2 Leray's Theorem

The examples we've seen so far ($H^1(\underline{X}, \mathcal{E}^{(k)})$ for $k = 0, 1, (1,0), (0,1), 2$ & $H^1(X, \underline{\mathbb{C}}_{\mathbb{Z}})$ for X simply connected) illustrate how to check if $H^1(X, \mathcal{F}) = 0$: show it for $H^1(\underline{\mathcal{U}}, \mathcal{F})$ for $\underline{\mathcal{U}}$ arbitrary open covering.

Q: what to do when $H^1(X, \mathcal{F}) \neq 0$?

Leray's Theorem will allow us to compute it as $H^1(\mathcal{U}, \mathcal{F})$ from a Leray covering. It works for any topological space (not just R.S.!)

Definition: Given X top space, $\mathcal{F}$ sheaf on X & $\underline{\mathcal{U}} = (U_i)_{i \in I}$ an open covering of X, we say $\underline{\mathcal{U}}$ is a Leray covering (of 1st order) for $\mathcal{F}$ if

$$H^1(\mathcal{U}_i, \mathcal{F}) = 0 \quad \forall i$$

(In general, for p^{th} order, we need $H^q(\mathcal{U}_i, \mathcal{F}) = 0 \quad \forall q \in \{1, \dots, p\}$ & $\forall i$)

Theorem (Leray): Fix a topological space X, a sheaf of ab gps/vector sp $\mathcal{F}$ on X and an open covering $\mathcal{U} = (U_i)_{i \in I}$ on X that is Leray for $\mathcal{F}$. Then,

$$H^1(X, \mathcal{F}) \simeq H^1(\underline{\mathcal{U}}, \mathcal{F}).$$ (iso wrt structure of $\mathcal{F}$)

Proof Fix $\mathcal{V}$ refinement of $\mathcal{U}$. Write $\underline{\mathcal{V}} = (V_k)_{k \in K}$ & pick $\tau: K \longrightarrow I$

Since

$$\begin{tikzcd} H^1(\underline{\mathcal{U}}, \mathcal{F}) \arrow[r] \arrow[d, hook, "\tau^{\mathcal{U}}_{\mathcal{V}}"'] & H^1(X, \mathcal{F}) \\ H^1(\underline{\mathcal{V}}, \mathcal{F}) \arrow[ur] & \end{tikzcd}$$

it's enough to show $\tau^{\mathcal{U}}_{\mathcal{V}}: H^1(\underline{\mathcal{U}}, \mathcal{F}) \xrightarrow{\simeq} H^1(\underline{\mathcal{V}}, \mathcal{F})$ is an iso if $\underline{\mathcal{U}}$ is Leray. By Theorem §16.2, it's enough to show r is surjective

• Pick $f \in Z^1(\underline{\mathcal{V}}, \mathcal{F})$.

<u>GOAL:</u> Find $g \in Z^1(\underline{\mathcal{U}}, \mathcal{F})$ with $\tau^{\mathcal{U}}_{\mathcal{V}}(\underline{g}) - \underline{f} \in B^1(\underline{\mathcal{V}}, \mathcal{F})$, ie,
$\exists\; h \in C^0(\underline{\mathcal{V}}, \mathcal{F})$ with $\tau^{\mathcal{U}}_{\mathcal{V}}(g) = f + \partial h$ on $V_{k_0 k_1}$ $\forall\, k_0, k_1 \in K^2$.

We build g by working on each $U_{i_0 i_1}$ & using the covering $(U_{i_0 i_1} \cap V_k)_k$.

Consider the open covering of U_i given by $\underline{W}^{(i)} = \{U_i \cap V_k\}_k$.

Since $H^1(U_i, \mathcal{F}) = 0$ $\forall i$, we know $H^1(\underline{W}^{(i)}, \mathcal{F}) = 0$

In particular $\left(f_{k_0 k_1}\big|_{U_i \cap V_{k_0} \cap V_{k_1}}\right) \in \mathcal{F}(U_i \cap V_{k_0} \cap V_{k_1})$ satisfies the cocycle equation (inherited from $f \in Z^1(\underline{\mathcal{V}}, \mathcal{F})$),

Thus, $\left(f_{k_0 k_1}|_{U_i \cap V_{k_0 k_1}}\right)_{k_0 k_1} \in Z^1(\underline{W}, \mathcal{F}) = B^1(\underline{W}, \mathcal{F})$

Pick $(\varphi^{(i)}_j)_j \in C^0(\underline{W}, \mathcal{F})$ with $f_{k_0 k_1}\big|_{U_i \cap V_{k_0 k_1}} = \varphi^{(i)}_{k_1} - \varphi^{(i)}_{k_0}$ on $U_i \cap V_{k_0} \cap V_{k_1}$
$(\varphi^{(i)}_k \in \mathcal{F}(U_i \cap V_k)\ \forall k \in K)$

For each fixed pair (i_0, i_1) we have

$$\boxed{f_{k_0 k_1}} = \varphi^{(i_0)}_{k_1} - \varphi^{(i_0)}_{k_0} = \varphi^{(i_1)}_{k_1} - \varphi^{(i_1)}_{k_0} \text{ on } U_{i_0 i_1} \cap V_{k_0 k_1} \ \forall k_0, k_1$$

$$\varphi^{(i_0)}_{k_1} - \varphi^{(i_1)}_{k_1} = \varphi^{(i_0)}_{k_0} - \varphi^{(i_1)}_{k_0} \text{ on } (U_{i_0 i_1} \cap V_{k_0}) \cap (U_{i_0 i_1} \cap V_{k_1})$$

By the sheaf axiom, these glue to a ! section $g_{i_0 i_1} \in \mathcal{F}(U_{i_0 i_1})$
with $\boxed{g_{i_0 i_1}|_{U_{i_0 i_1} \cap V_k} = \varphi^{(i_0)}_k - \varphi^{(i_1)}_k \quad \forall k \in K}$.

<u>Claim 1</u>: $(g_{i_0 i_1})_{i_0 i_1} \in Z^1(\underline{\mathcal{U}}, \mathcal{F})$

Pf/ $\partial g|_{i_0 i_1 i_2} = g_{i_1 i_2} - g_{i_0 i_2} + g_{i_0 i_1} \overset{?}{=} 0$ on $U_{i_0 i_1 i_2}$

We check it on $U_{i_0 i_1 i_2} \cap V_k$ for $k \in K$

$$(\text{LHS})\big|_{U_{i_0 i_1 i_2} \cap V_k} = \underbrace{\varphi_k^{(i_1)} - \varphi_k^{(i_2)}}_{= g_{i_1 i_2}} - \underbrace{(\varphi_k^{(i_0)} - \varphi_k^{(i_2)})}_{g_{i_0 i_2}} + \underbrace{\varphi_k^{(i_0)} - \varphi_k^{(i_1)}}_{g_{i_0 i_1}} = 0 \quad \text{on } U_{i_0 i_1 i_2} \cap V_K$$

$\square$

- Next, we need to find $h \in C^0(\underline{V}, \mathcal{F})$ so

$$g_{\tau k_0 \tau k_1} - f_{k_0 k_1} = (\partial h)_{k_0 k_1} = h_{k_1} - h_{k_0} \quad \text{on } V_{k_0 k_1}$$

<u>Claim 2:</u> $h_k = -\varphi_k^{(\tau k)} \in \mathcal{F}(V_k)$ satisfies the conditions.

Pf/ On $U_{\tau k_0 \tau k_1} \cap V_{k_0 k_1} = V_{k_0 k_1}$ we have

$$g_{\tau k_0 \tau k_1} = \varphi_{k_0}^{(\tau k_0)} - \varphi_{k_0}^{(\tau k_1)} = \varphi_{k_1}^{(\tau k_0)} - \varphi_{k_1}^{(\tau k_1)} \quad \& \quad f_{k_0 k_1} = \varphi_{k_1}^{(\tau k_0)} - \varphi_{k_0}^{(\tau k_0)} = \varphi_{k_1}^{(\tau k_1)} - \varphi_{k_0}^{(\tau k_1)}$$

$$\Rightarrow g_{\tau k_0 \tau k_1} - f_{k_0 k_1} = \varphi_{k_1}^{(\tau k_0)} - \varphi_{k_1}^{(\tau k_1)} - (\varphi_{k_1}^{(\tau k_0)} - \varphi_{k_0}^{(\tau k_0)}) = \varphi_{k_0}^{(\tau k_0)} - \varphi_{k_1}^{(\tau k_1)}$$

$$= h_{k_1} - h_{k_0} = (\partial h)_{k_0 k_1} \quad \text{on } V_{k_0 k_1}.$$

$\square$

For Riemann surfaces, this will be enough, but the statement is more general:

<u>General Leray's Theorem:</u> Given X top space, $\mathcal{F}$ sheaf on X & a Leray covering $\underline{\mathcal{U}}$ of order p for $\mathcal{F}$ ($H^q(\mathcal{U}_i, \mathcal{F}) = 0 \quad \forall\, 1 \leq q \leq p \quad \forall i \in I$), then

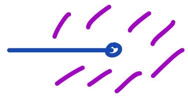
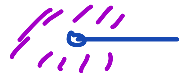
$$H^p(\underline{\mathcal{U}}, \mathcal{F}) \simeq H^p(X, \mathcal{F}).$$

<u>Application:</u> $H^1(\mathbb{C}^*, \underline{\mathbb{Z}}) = \mathbb{Z}$ & $H^1(\mathbb{C}^*, \underline{\mathbb{C}}) = \mathbb{C}$

<u>Proof</u>: We set $\mathcal{F} = \underline{\mathbb{Z}}$ or $\underline{\mathbb{C}}$ & write $R = \mathbb{Z}$ or $\mathbb{C}$ for the corresponding ring

We need to find a Leray open cover for $\mathbb{C}^\times$ (enough: use simply connected sets by Theorem §17.1)

$\mathbb{C}^\times = \mathcal{U}_0 \cup \mathcal{U}_1$ with $\mathcal{U}_0 = \mathbb{C}^\times \setminus \mathbb{R}_{<0}$, $\mathcal{U}_1 = \mathbb{C}^\times \setminus \mathbb{R}_{>0}$

- Both $\mathcal{U}_0$ & $\mathcal{U}_1$ are simply connected, so by Theorem 4 $H^1(\mathcal{U}_0, \mathbb{Z}) = H^1(\mathcal{U}_1, \mathbb{Z}) = 0$
- There are no triple intersections, so $Z^1(\underline{\mathcal{U}}, \mathcal{F}) = C^1(\underline{\mathcal{U}}, \mathcal{F}) = \mathcal{F}(U_0 \cap U_1)$

  $(C^2(\underline{\mathcal{U}}, \mathcal{F}) = \{0\})$

$U_0 \cap U_1 = \mathbb{C} \setminus \mathbb{R}$ has 2 connected components (upper & lower half plane)

So $Z^1(\underline{\mathcal{U}}, \mathcal{F}) \simeq \mathbb{R} \oplus \mathbb{R}$

$\underline{Q}$: What is $B^1(\underline{\mathcal{U}}, \mathcal{F}) \leq Z^1(\underline{\mathcal{U}}, \mathcal{F})$? $C^0(\underline{\mathcal{U}}, \mathcal{F}) = \mathcal{F}(U_0) \oplus \mathcal{F}(U_1) = \mathbb{R}^2$

The coboundary operator $\partial : C^0(\underline{\mathcal{U}}, \mathcal{F}) \longrightarrow B^1(\underline{\mathcal{U}}, \mathcal{F}) \leq Z^1(\underline{\mathcal{U}}, \mathcal{F})$

$$(h_0, h_1) \longrightarrow (h_1 - h_0) \in \mathcal{F}(U_0 \cap U_1)$$

On each component of $U_0 \cap U_1$ it takes the same value, so we view

$B^1(\underline{\mathcal{U}}, \mathcal{F}) \leq Z^1(\underline{\mathcal{U}}, \mathcal{F}) \simeq \mathbb{R}^2$ as the diagonal.

Thus $H^1(\underline{\mathcal{U}}, \mathcal{F}) = \mathbb{R}^2 / \mathbb{R}(1,1) \cong \mathbb{R}$

By Leray's Theorem : $H^1(\underline{\mathcal{U}}, \mathcal{F}) \simeq H^1(X, \mathcal{F})$, so we are done $\square$

Next topic: Proof idea of finite dimensionality of $H^1(X, \mathcal{O})$ for X compact R.S

§17.3 $H^1(D_R(0), \mathcal{O}) = H^1(\mathbb{C}, \mathcal{O}) = H^1(\mathbb{P}^1, \mathcal{O}) = 0$

In order to show $H^1(X, \mathcal{O}) = 0$ for $X = D_R(0)$, $\mathbb{C}$ & $\mathbb{P}^1$ we'll need the following statement (see Lecture 18 for a proof sketch) of Dolbeault.

Notation: $\mathbb{C} = D_\infty(0)$.

Dolbeault's Theorem: Fix $X = D_R(0) = \{z : |z| < R\}$ for $0 < R \leq \infty$ & $g \in \mathcal{E}(X)$

Then $\exists f \in \mathcal{E}(X)$ with $\frac{\partial f}{\partial \bar{z}} = g$

Theorem: Suppose $X = D_R(0)$ for $0 < R \leq \infty$. Then $H^1(X, \mathcal{O}) = 0$.

Proof: Take an arbitrary open cover $\underline{\mathcal{U}} = \{U_i\}_{i \in I}$ & a cocycle $(g_{ij}) \in Z^1(\underline{\mathcal{U}}, \mathcal{O})$

Recall $H^1(X, \mathcal{E}) = 0$ & $Z^1(X, \mathcal{O}) \subseteq Z^1(X, \mathcal{E})$ so $(g_{ij}) \in B^1(X, \mathcal{E})$

Pick $f \in \mathcal{E}(X)$ with $g_{ij} = f_j - f_i$ on U_{ij}.

But $g_{ij} \in \mathcal{O}(U_{ij})$ so $\bar{\partial} g_{ij} := \frac{\partial g_{ij}}{\partial \bar{z}} = 0$

$\Rightarrow \bar{\partial} f_j - \bar{\partial} f_i = 0$ on U_{ij} so by the sheaf axiom, the sections $\bar{\partial} f_i$ on $\mathcal{E}(U_i)$ glue to a global function h on $\mathcal{E}(X)$ with $h|_{U_i} = \bar{\partial} f_i$

By Dolbeault's Theorem §17.9, we can find $F \in \mathcal{E}(X)$ with $\frac{\partial F}{\partial \bar{z}} = h$

Define: $G_i = f_i - F$ on $\mathcal{E}(U_i)$

Now:
- $\frac{\partial G_i}{\partial \bar{z}} = \frac{\partial f_i}{\partial \bar{z}} - \frac{\partial F}{\partial \bar{z}} = h|_{U_i} - h|_{U_i} = 0$ on $U_i \Rightarrow G_i \in \mathcal{O}(U_i)$
- $G_i = f_i - F = f_j - F = G_j$ on U_{ij} so G_i lift to $G \in \mathcal{O}(X)$
- $g_{ij} = f_j - f_i = (f_j - F) - (f_i - F) = G_j - G_i$ so $g_{ij} = \partial G_{ij}$

Conclude $(g_{ij}) \in B^1(X, \mathcal{O})$ so $H^1(X, \mathcal{O}) = 0$. $\square$

Corollary: $H^1(\mathbb{P}^1, \mathcal{O}) = 0$

Proof We show $H^1(\underline{\mathcal{U}}, \mathcal{O}) = 0$ for $\underline{\mathcal{U}} = \{\underbrace{\mathbb{P}^1 \setminus \{0\}}_{U_\infty}, \underbrace{\mathbb{P}^1 \setminus \{\infty\}}_{U_0}\}$

Advantage: There are no triple intersections, so the Čech complex is finite.

$$0 \longrightarrow C^0(\underline{U},\mathcal{O}) \longrightarrow C^1(\underline{U},\mathcal{O}) \longrightarrow 0 = C^2(\underline{U},\mathcal{O}).$$

Claim: $H^1(\underline{U},\mathcal{O}) = 0$

Pf/ $C^1(\underline{U},\mathcal{O}) = Z^1(\underline{U},\mathcal{O}) = \mathcal{O}(U_0 \cap U_1) = \mathcal{O}(\mathbb{C}^*)$

Any $f \in \mathcal{O}(\mathbb{C}^*)$ has a power series expansion:

$$f = \sum_{n=-\infty}^{\infty} f_n z^n = \sum_{n=0}^{\infty} f_n z^n + \sum_{n=-\infty}^{-1} f_n z^n$$

$$= \underbrace{\sum_{n=0}^{\infty} f_n z^n}_{\in \mathcal{O}(U_0)} - \underbrace{\sum_{n=1}^{\infty} (-f_{-n}) z^{-n}}_{\in \mathcal{O}(U_\infty)}$$

So $f \in B^1(\underline{U},\mathcal{O})$. $\square$