Lecture XVIII: Integration of 2-forms & Dolbeault's Thm

Last time, we used Dolbeault's Thm to show $H^1(D_R(0), \mathcal{O}) = H^1(\mathbb{C}, \mathcal{O}) = 0$.

Notation: $\mathbb{C} = D_\infty(0)$

Dolbeault's Theorem: Fix $X = D_R(0) = \{z : |z| < R\}$ for $0 < R \leq \infty$ & $g \in \mathcal{E}(X)$

Then $\exists f \in \mathcal{E}(X)$ with $\dfrac{\partial f}{\partial \bar{z}} = g$

Today, we'll give a proof sketch of this result. We'll build f by an integral.

For this, we need to talk about integration of differentiable 2-forms.

§18.1. Integration of 2-forms on $\mathbb{C}$:

As usual, we start with the case $X = \mathbb{C}$. Fix $U \subseteq \mathbb{C}$ open & $\omega \in \mathcal{E}^{(2)}(U)$.

Write $\omega = f \, dx \wedge dy$ with $f \in \mathcal{E}(U)$

Def: Assume f has compact support (ie $\exists K \subseteq U$ compact st $f|_{U \smallsetminus K} \equiv 0$). Then

$$\iint_U \omega := \iint_U f_{(x,y)} \, dx \, dy$$

- Next, we want to see what happens under holomorphic coordinate changes:

Lemma: Fix $V \subseteq \mathbb{C}$ open & $\underset{(u,v)}{\varphi : V} \longrightarrow \underset{(x,y)}{U}$ a biholomorphic map. If $\omega \in \mathcal{E}^{(2)}(U)$, then

$$\iint_{U = \varphi(V)} \omega = \iint_V \varphi^* \omega$$

Proof: Write $\varphi = x + iy \Rightarrow \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$

This gives $J_{(x,y)} := \det\left(\dfrac{\partial(x,y)}{\partial(u,v)}\right) = x_u y_v - x_v y_u \underset{CR}{=} x_u^2 + x_v^2 = |\varphi'|^2$ $\dfrac{\partial \varphi}{\partial z}$

The classical change of coordinates gives $\iint_U f\, dx\, dy = \iint_V f\circ\varphi \cdot J_{(x,y)}\, du\, dv$

But $\varphi^*(dz\wedge d\bar z) = d\varphi \wedge d\bar\varphi \underset{\substack{\varphi \text{ holo}, \\ \bar\varphi \text{ antiholo}}}{=} \left(\frac{\partial \varphi}{\partial z} dz\right) \wedge \left(\underbrace{\frac{\partial \bar\varphi}{\partial \bar z}}_{\overline{\frac{\partial \varphi}{\partial z}} \text{ (Lecture 14)}} d\bar z\right) = \varphi' \cdot \overline{\varphi'}\, dz\wedge d\bar z$

$= |\varphi'|^2 dz\wedge d\bar z$

$\Rightarrow \varphi^*\omega = f\circ\varphi \cdot |\varphi'|^2\, du\wedge dv \in \mathcal{E}^{(2)}(V)$ gives $\iint_U \omega = \iint_V \varphi^*\omega$. $\square$

§18.2. Integration of 2-forms on R.S.:

Fix X RS. To define integration of 2-forms we restrict to compact support.

Def: Given $\omega \in \mathcal{E}^{(2)}(X)$, the support of ω is defined as

$$\operatorname{Supp}(\omega) := \overline{\{a\in X : \omega_{(a)} \neq 0\}}$$

To define $\iint_X \omega$ with $\operatorname{Supp}(\omega)$ compact, we proceed in 2 steps:

1. we restrict to local charts & compactly supported scalars
2. we use partition of unity to extend from 1 to a general compactly supported ω

STEP 1: Fix K (compact) support of ω & assume we have (U,φ) local chart with $K\subseteq U$. Write $\varphi: U \xrightarrow{\sim} \underset{\text{open}}{V} \subseteq \mathbb{C}$, so $(\varphi^{-1})^*: \mathcal{E}^{(2)}(U) \to \mathcal{E}^{(2)}(V)$

Then $\iint_X \omega = \iint_U \omega := \iint_V (\varphi^{-1})^*\omega$

Lemma 1: The definition is indep of the choice of charts

Proof: Assume $\psi: U_1 \xrightarrow{\sim} V_1$ is another choice of charts with $K\subseteq U_1$.

We can restrict to $U = U_1$ (otherwise take $U\cap U_1 = U'$ & $\begin{matrix} V' = \varphi(U') \\ V_1' = \psi(U') \end{matrix}$).

Consider $V_1 \xrightarrow{\varphi^{-1}} U \xrightarrow{\psi} V$ biholomorphism.

By Lemma §18.1 $\overset{\omega'}{\omega'} = (\varphi^{-1})^* \overset{\omega}{\omega}$ satisfies $\iint_V \omega' = \iint_{V_1} (\psi \circ \varphi^{-1})^* \omega'$

But $(\psi \circ \varphi^{-1})^* \omega' = (\varphi^{-1})^* \circ \psi^* \omega' = (\varphi^{-1})^* \circ \psi^* \circ (\psi^{-1})^* \omega = (\varphi^{-1})^* \omega$

So $\iint_V (\psi^{-1})^* \omega = \iint_V \omega' = \iint_{V_1} (\varphi^{-1})^* \omega$, as we wanted to show.

STEP 2: We cover K with finitely many local charts $(U_K, U_K \xrightarrow[\sim]{\varphi_K} V_K \underset{\text{open}}{\subseteq} \mathbb{C})$

We use a partition of unity $(f_i)_{1 \leq i \leq n}$ subordinate to $\{(U_K, \varphi_K)\}_{K=1}^{n}$ to define $\iint \omega$.

Definition: $\iint_X \omega = \iint_K \omega = \sum_{i=1}^{n} \iint_{U_i} f_i \, \omega$

Lemma 2: This is well-defined (ie, indep of choice of open covering & partition of 1)

Proof: It's enough to show that the statement holds for refinements (take common refinement of 2 covers & the associated partition of unity).

This is easy to check by covering open charts & showing the definition of ① agrees with ② in this special case. All our sums are finite so we can manange at will.

§18.3 Stokes' Theorem:

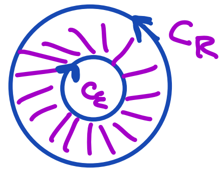
Next, we formulate a special case of Stokes' Thm in the plane.

Theorem 1: Fix $U \subseteq \mathbb{C}$ open & $A \subseteq U$ compact set with ∂A smooth. We orient ∂A so that the outward normal & the tangent vector form a positively oriented basis for $\mathbb{C}$ (ie orientation inherited from $\mathbb{C}$). Then, for any $\omega \in \mathcal{E}^{(1)}(U)$, we have $\iint_A d\omega = \int_{\partial A} \omega$.

We will only need the case when A is a disk or an annulus, ie.

$$A = \{ z \in \mathbb{C} : \varepsilon \leq |z| \leq R \} \quad \text{for } 0 < \varepsilon < R.$$

If $\omega = f\,dx + g\,dy$, we have $d\omega = (g_x - f_y)\,dx \wedge dy$



$\partial A = C_R \cup C_\varepsilon$

$A = \overline{D}_R - D^\circ_\varepsilon$.

$$(*) \quad \iint_{\varepsilon \le |z| \le R} (g_x - f_y)\,dx\,dy = \int_{C_R} (f\,dx + g\,dy) - \int_{C_\varepsilon} f\,dx + g\,dy$$

because of orientation

Proof for annulus: We work with polar coordinates.

Set $z = re^{i\theta}$ $\quad \varepsilon \le r \le R \quad 0 \le \theta \le 2\pi \quad x = r\cos\theta$, $y = r\sin\theta$

$$r = \sqrt{x^2+y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \frac{dr}{dx} = \frac{x}{r} = \cos\theta$$

$$\frac{d\theta}{dx} = \frac{1}{1+\frac{y^2}{x^2}} \frac{-y}{x^2} = \frac{-y}{x^2+y^2} = \frac{-\sin\theta}{r}$$

It's enough to show it for $\omega = g\,dy$ & $\omega = f_x$ separately, and use linearity.

STEP 1: $\omega = g\,dy$ $\quad$ Write $g(x,y) = G(r,\theta)$

$$\Rightarrow g_x\,dx \wedge dy = \left(\frac{\partial G}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial G}{\partial \theta}\frac{\partial \theta}{\partial x}\right)\left(\frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta\right) \wedge \left(\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta\right)$$

$$= \left(G_r\cos\theta + G_\theta\left(\frac{-\sin\theta}{r}\right)\right)(\cos\theta\,dr - r\sin\theta\,d\theta) \wedge (\sin\theta\,dr + r\cos\theta\,d\theta)$$

$$= \left(G_r\cos\theta - G_\theta\frac{\sin\theta}{r}\right)(r\cos^2\theta + r\sin^2\theta)\,dr \wedge d\theta$$

$$= (G_r r\cos\theta - G_\theta\sin\theta)\,dr \wedge d\theta = \left(\cos\theta\frac{\partial}{\partial r}(rG) - \frac{\partial}{\partial \theta}(\sin\theta\,G)\right)dr \wedge d\theta$$

(LHS) of (*) becomes:

$$\iint_{\varepsilon < |z| \le R} g_x\,dx\,dy \underset{\text{change of vars}}{=} \int_0^{2\pi}\int_\varepsilon^R \cos\theta\frac{\partial}{\partial r}(rG) - \frac{\partial}{\partial \theta}(\sin\theta\,G)\,dr\,d\theta$$

BUT $\displaystyle\int_0^{2\pi}\int_\varepsilon^R -\frac{\partial}{\partial \theta}(\sin\theta\,G)\,dr\,d\theta = \int_\varepsilon^R \sin\theta\,G(r,\theta)\Big|_0^{2\pi} dr = 0$, so

only the first term survives:

$$\iint_{\varepsilon < |z| \le R} g_x\,dx\,dy = \int_0^{2\pi} \cos\theta\,(RG(R,\theta) - \varepsilon G(\varepsilon,\theta))\,d\theta = (\text{RHS}) \text{ of } (*)$$

because $\displaystyle\int_{C_r} g\,dy = \int_0^{2\pi} G\cdot r\cos\theta\,d\theta \qquad \forall\ \varepsilon\le r\le \mathcal{E}.$

($g\,dy = G\,r\cos\theta\,d\theta$ if r is fixed.)

STEP 2 $\quad \omega = f\,dx \quad$ Use the change of coordinates

$(x,y) \longrightarrow -(y,x)$ (we need the $-$ to preserve the orientation)

This transformation has determinant 1 so

$$\iint_A f_y\,dx\,dy = \iint_A \tilde{g}_x\,dx\,dy$$

$$\int_{\partial A} f\,dx = \int_{\partial A} \tilde{g}\,dy$$

Proof for the disc: Let $\varepsilon\to 0^+$ for the annulus & notice $\displaystyle\int_{C_\varepsilon} f\,dx+p\,dy \xrightarrow[\varepsilon\to 0^+]{} 0$

Theorem 2: Suppose X is a R.S & $\omega\in\mathcal{E}^{(1)}(X)$ is a differential form with compact support. Then $\displaystyle\iint_X d\omega = 0$

Proof: Fix U open with $U\supset \operatorname{Supp} X = K$ compact. Find local charts $(U_i,\varphi_i)_{i=1}^N$ covering K inside U & pick a partition of unit subordinate to it. $\{f_1,\dots,f_N\}$

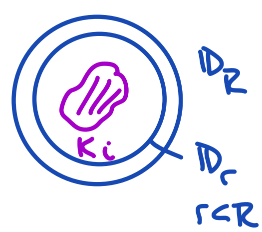
Write $\omega_i = f_i\,\omega$ so $\omega = \omega_1+\dots+\omega_N$.

Then each ω_i has compact support & $\operatorname{Supp}\omega_i \subseteq U_i \quad \forall i=1,\dots,N$

In this setting, we are reduced to $X=\mathbb{C}$, $U_i=\mathbb{D}_R \supset \operatorname{Supp}(\omega_i)$

Then $$\iint_{\mathbb{C}} d\omega_i = \iint_{|z|\le R} d\omega_i = \int_{|z|=R}\omega_i = \int_{|z|=R} 0 = 0.$$

$\operatorname{Supp}\omega_i \cap \{|z|=R\} = 0$



(since $K_i := \operatorname{Supp}\omega_i \subset \mathbb{D}_R$ is compact, $\exists\ 0\le r<R$ with $K_i\subseteq \mathbb{D}_r$)

§18.4 Dolbeault's Lemma:

Our next objective is to confirm $\dim_{\mathbb{C}} H^1(X,\mathcal{O}) < \infty$ whenever X is a compact RS

We'll show this for $X = \mathbb{P}^1$. In order to do this, we need to solve the equation

$$\frac{\partial f}{\partial \bar{z}} = g \quad \text{for } f, g \in \mathcal{E}(X) \qquad \text{for } X = \mathbb{C} \text{ or } D_R(0) \subseteq \mathbb{C}$$

(Name: Inhomogeneous Cauchy-Riemann Eqn)

Lemma (Dolbeault): Assume $g \in \mathcal{E}(\mathbb{C})$ has compact support. Then $\exists f \in \mathcal{E}(\mathbb{C})$ with $\dfrac{\partial f}{\partial \bar{z}} = g$

Proof The solution f is given by a Cauchy-Type integral

$$f(\zeta) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{g(z)}{\zeta - z}\, dz \wedge d\bar{z}. = \frac{1}{2\pi i} \iint_{D_R(\zeta)} \frac{g(z)}{\zeta - z}\, dz \wedge d\bar{z}$$

if $\operatorname{Supp}(g) \subseteq \overline{D_R(\zeta)}$

Claim 1: $f \in \mathcal{E}(\mathbb{C})$

We compute $\iint$ in polar coordinates. Write
$$z = \zeta + re^{i\theta} = \zeta + x + iy$$
$$\bar{z} = \bar{\zeta} + re^{-i\theta}$$

$$dz \wedge d\bar{z} = (dx + i\,dy) \wedge (dx - i\,dy) = -2i\, dx \wedge dy$$

But $x = r\cos\theta$, $y = r\sin\theta$, so $dx = \cos\theta\, dr - r\sin\theta\, d\theta$, $dy = \sin\theta\, dr + r\cos\theta\, d\theta$

$$\Rightarrow dz \wedge d\bar{z} = -2i\,(r\cos^2\theta + r\sin^2\theta)\, dr \wedge d\theta = -2ir\, dr \wedge d\theta$$

Thus
$$f(\zeta) = \frac{1}{2\pi i} \iint_{\substack{0 \le r \le R \\ 0 \le \theta < 2\pi}} \frac{g(\zeta + re^{i\theta})}{re^{i\theta}} (-2ir)\, dr\, d\theta = -\frac{1}{\pi} \iint \underbrace{g(\zeta + re^{i\theta})\, e^{-i\theta}}_{\in \mathcal{E}([0,R]\times[0,2\pi])} dr\, d\theta$$

This shows f is well-defined & it's a smooth function. $\square$

• __Claim 2__: $\frac{\partial f}{\partial \bar\zeta} = g$

We compute partial derivatives under the sign integral

$$\frac{\partial f}{\partial \bar\zeta} = \frac{-1}{\pi} \int_0^{2\pi}\int_0^R \frac{\partial g}{\partial \bar\zeta}(\zeta + re^{i\theta})\, e^{-i\theta} \underbrace{dr\, d\theta}_{= \frac{dz\wedge d\bar z}{-2ir}}$$

$$= \frac{1}{2\pi i} \lim_{\varepsilon\to 0^+} \iint_{B_\varepsilon} \frac{\partial g}{\partial \bar\zeta}(\zeta+z)\, \frac{dz\wedge d\bar z}{z} \qquad B_\varepsilon = \{z : \varepsilon \leq |z| \leq R\}$$

reverse the polar coord expression $z = re^{i\theta}$

Here, we used the fact that the first integral is well-defined to express it as a limit of an integral defined away from $z=0$.

Now: $\frac{\partial g}{\partial \bar\zeta}(\zeta+z)\frac{1}{z} = \frac{\partial g}{\partial \bar z}(\zeta+z)\cdot\frac{1}{z} = \frac{\partial}{\partial \bar z}\left(g(\zeta+z)/z\right)$ for $z\neq 0$

So $\frac{\partial f}{\partial \bar\zeta} = \frac{1}{2\pi i} \lim_{\varepsilon\to 0^+} \iint_{B_\varepsilon} \frac{\partial}{\partial \bar z}\left(g(\zeta+z)/z\right) dz\wedge d\bar z = -\lim_{\varepsilon\to 0^+} \iint_{B_\varepsilon} d\omega$

where $\omega(z) = \frac{1}{2\pi i}\frac{g(\zeta+z)}{z}\, dz$ (ζ is a constant & z is the variable)

By Stoke's Thm on B_ε we get

$$\frac{\partial f}{\partial \bar\zeta} = -\lim_{\varepsilon\to 0^+} \iint_{B_\varepsilon} d\omega = -\lim_{\varepsilon\to 0^+} \iint_{\partial B_\varepsilon} \omega = -\lim_{\varepsilon\to 0^+}\left(\int_{C_R}\omega - \int_{C_\varepsilon}\omega\right)$$

$$= -\int_{C_R}\omega + \lim_{\varepsilon\to 0^+}\int_{C_\varepsilon}\omega$$

If $R \gg 1$, then $g=0$ on C_R so $\int_{C_R}\omega = 0$.

Now $\frac{dz}{z} = \frac{d(\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} = \frac{e^{i\theta}d\varepsilon}{\varepsilon e^{i\theta}} + \frac{i\varepsilon e^{i\theta}d\theta}{\varepsilon e^{i\theta}} = i\,d\theta$ on C_ε (ε is constant)

$$\Rightarrow \int_{C_\varepsilon}\omega = \int_0^{2\pi} \frac{1}{2\pi i} g(\zeta+\varepsilon e^{i\theta})\, i\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} g(\zeta+\varepsilon e^{i\theta})\, d\theta$$

Conclude $\frac{\partial f}{\partial \bar{\xi}} = \lim_{\varepsilon \to 0^+} \int_{C_\varepsilon} \omega = \frac{1}{2\pi} \underbrace{\lim_{\varepsilon \to 0^+} \int_0^{2\pi} g(\xi + \varepsilon e^{i\theta})\, d\theta}_{= g(\xi)\, 2\pi \quad (\varepsilon/\delta \text{ argument})} = g(\xi)$

$\square$

We can remove the compact support condition if we work with $X = D_R(0)$. $(R \in (0, \infty])$

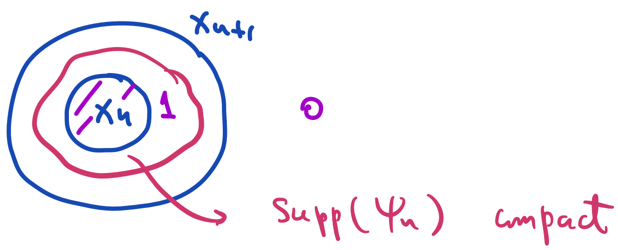
The next result is a special case of Dolbeault's Lemma in several complex variables.

Dolbeault's Theorem: Fix $X = D_R(0) = \{z : |z| < R\}$ for $0 < R \leq \infty$ & $g \in \mathcal{E}(X)$

Then $\exists f \in \mathcal{E}(X)$ with $\frac{\partial f}{\partial \bar{z}} = g$

Proof Take a sequence $R_0 < R_1 < \cdots$ with $\lim_{n \to \infty} R_n = R$ & set $X_n := D_{R_n}(0)$

Take bump functions $\psi_n \in \mathcal{E}(X)$ with
- $\operatorname{supp}(\psi_n) \subseteq X_{n+1}$ compact
- $\psi_n|_{X_n} \equiv 1$



$\operatorname{Supp}(\psi_n) \cap \partial X_{n+1} = \emptyset$ since X_n open & $\operatorname{Supp}(X_n)$ is cpt.

By Dolbeault's Lemma applied to $g_n = \psi_n g \in \mathcal{E}(X_n)$ $\exists f_n \in \mathcal{E}(X)$ with $\frac{\partial f_n}{\partial \bar{z}} = \psi_n g$ on X_n

By induction on $n \in \mathbb{N}_0$ we build $h_0, h_1, h_2, \ldots$ st

1. $\frac{\partial h_n}{\partial \bar{z}} = g|_{X_n}$ on X_n
2. $\sup_{x \in \bar{X}_{n-1}} |h_{n+1}(x) - h_n(x)| = \| h_{n+1} - h_n \|_{X_{n-1}} \leq 2^{-n}$ (uniform convergence condition)

(Idea $h_0 = f_0$, $h_n = f_n - P$ for a suitable polynomial coming from the Taylor series expansion of $f_{n+1} - h_n$ about 0 with suitable bound on the tail).

- $f = \lim_{n\to\infty} h_n(z)$ & show $f|_{X_n} = h_n + \underbrace{\sum_{k=n}^{\infty} (h_{n+1} - h_k)}_{\text{holomorphic}}$ on X_n

$\Rightarrow f|_{X_n} \in \mathcal{E}(X_n)\ \forall n$ & $\dfrac{\partial f}{\partial \bar z} = g|_{X_n}$ on X_n. $\square$