Lecture XIX: Finiteness of $\dim_{\mathbb{C}} H^1(X,\mathcal{O})$ for X compact

Recall Čech cohomology as $H^\bullet(X,\mathcal{F}) = \varinjlim_{\underline{\mathcal{U}}} H^\bullet(\underline{\mathcal{U}},\mathcal{F})$ $\quad$ ($H^1(\underline{\mathcal{U}},\mathcal{F}) \hookrightarrow H^1(\mathcal{V},\mathcal{F})$ if $\mathcal{V}<\mathcal{U}$ is refinement)

(X top space) $H^1(X,\mathcal{F})=0 \iff \forall\, \underline{\mathcal{U}}$ open covering $H^1(\underline{\mathcal{U}},\mathcal{F})=0$

- **Leray's Thm:** If $\underline{\mathcal{U}}$ is Leray ($H^1(\mathcal{U}_i,\mathcal{F})=0\ \forall i$) $\Rightarrow H^1(X,\mathcal{F}) \simeq H^1(\underline{\mathcal{U}},\mathcal{F})$

- **Examples:**
  1. $H^1(X,\underline{\mathbb{C}}) = H^1(X,\underline{\mathbb{Z}})=0$ if X is a simply connected R.S.
  2. $H^1(X,\mathcal{E}^{(k)})=0$ for $k=0,1,(1,0),(0,1),2$ for X R.S.
  3. $H^1(D_R(0),\mathcal{O}) = H^1(\mathbb{C},\mathcal{O}) = H^1(\mathbb{P}^1,\mathcal{O})=0$ $\quad (0<R<\infty)$

[Riemann Uniformization Theorem will say these are the only ones (up to biholomorphism)]

Key for 3 was Dolbeault's Theorem (see Lecture XVIII for a proof sketch).

Dolbeault's Theorem: Fix $X = D_R(0) = \{z : |z|<R\}$ for $0<R\leq\infty$ & $g\in\mathcal{E}(X)$ ($\mathbb{C}=D_\infty(0)$)

Then $\exists\, f\in\mathcal{E}(X)$ with $\dfrac{\partial f}{\partial \bar z} = g$

Next goal:
- Show $\dim_{\mathbb{C}}(X,\mathcal{O})<\infty$ for X compact R.S. Call it genus of X
- Build non-constant meromorphic functions on compact R.S. with various restrictions. (General case: Riemann-Roch Thm)

§19.1 Finiteness Theorem:

Finiteness Theorem: If X is a compact R.S., then $H^1(X,\mathcal{O})$ is finite dim'l

The result is a special case of:

Theorem: Fix X a RS, Y_1,Y_2 opens with $Y_1 \Subset Y_2 \subset X$ ($\overline{Y_1}\subset Y_2$ is compact). Then, the restriction map $H^1(Y_2,\mathcal{O}) \xrightarrow{\text{res}} H^1(Y_1,\mathcal{O})$ has finite dimensional image.

The proof is described in detail in §14 & 15 of Forster's textbook. We'll show where the main difficulty lies.

<u>Remark:</u> First, define res on Čech cohomology relative to coverings

Fix $\underline{\mathfrak{U}} = \{U_i\}_{i\in I}$ open covering of Y_2. Then $\underline{\mathfrak{V}} = \{U_i \cap Y_1\}$ is a covering of Y_1

Take $Z^1(\underline{\mathfrak{U}}, \mathcal{O}) \xrightarrow{\text{res}} Z^1(\underline{\mathfrak{V}}, \mathcal{O})$

$$f_{ij} \in \mathcal{O}(U_{ij}) \longmapsto (f_{ij}|_{U_{ij}\cap Y_1})$$

<u>Note</u> res commutes with ∂ (exercise), so it defines $\text{res}: H^1(\underline{\mathfrak{U}}, \mathcal{O}) \longrightarrow H^1(\underline{\mathfrak{V}}, \mathcal{O})$.

Note: res is compatible with refinements on covers of Y_1 & Y_2.

$\Rightarrow$ Take direct limit on $\underline{\mathfrak{V}}$ & use defining property of $\varinjlim_{\mathfrak{U}}$.

$$\begin{tikzcd}
H^1(\underline{\mathfrak{U}}, \mathcal{O}) \arrow[r, "\text{res}"] \arrow[dr] & H^1(\mathfrak{V}, \mathcal{O}) \arrow[r] \arrow[d, phantom, "\circlearrowleft"] & \varinjlim_{\mathfrak{V}} H^1(\mathfrak{V}, \mathcal{O}) = H^1(Y, \mathcal{O}) \\
& \varinjlim_{\underline{\mathfrak{U}}} H^1(\underline{\mathfrak{U}}, \mathcal{O}) = H^1(X, \mathcal{O}) \arrow[ur, "\text{res}"'] &
\end{tikzcd}$$

<u>Proof</u> Pick a nice enough open covering & prove the statement for it. $\square$

For each $y \in Y_2$, consider a local chart $U_y \underset{z}{\cong} \mathbb{D}$, $y \longmapsto 0$

Pick $W_y = z^{-1}(\mathbb{D}_{\frac{1}{2}}(0))$ for each $y \in Y_2$. Then, $\overline{W_y} \subseteq U_y$ is compact for each $y \in Y_2$

- The collection $\{W_y : y \in Y_2\}$ covers Y_2, hence also $\overline{Y_1} \subseteq Y_2$.

By compactness of Y_1, we can find a subcover $\{W_1, \dots, W_n\}$. Consider the corresponding opens $\{U_1, \dots, U_n\}$ $\quad (W_i = W_{y_i} \Rightarrow U_i = U_{y_i})$

Then $Y_1 \subset \bigcup_{i=1}^n W_i =: Y' \Subset Y'' := \bigcup_{i=1}^n U_i \subset Y_2 \quad (W_i \Subset U_i \ \forall i)$

Note: $\overline{Y'} = \bigcup_{i=1}^{n} \overline{W_i}$ is compact & $\overline{W_i}$ is a closed disc.

Take $\underline{\mathcal{U}} = \{U_i : 1 \le i \le n\}$ $\underline{\mathcal{W}} = \{W_i : 1 \le i \le n\}$ coverings of Y'' & Y' by opens homeomorphic to open discs (in $\mathbb{C}$)

*Key step: Show $H^1(\underline{\mathcal{U}}, \mathcal{O}) \xrightarrow{res} H^1(\underline{\mathcal{W}}, \mathcal{O})$ has finite dim'l image (uses a lot of analysis & L^2 norms on cochains)

By construction, $\underline{\mathcal{U}}$ & $\underline{\mathcal{W}}$ are Leray coverings (opens are discs + use Theorem §17.5)

$$\begin{tikzcd}
H^1(\underline{\mathcal{U}}, \mathcal{O}) \arrow[r, "res"] & H^1(\underline{\mathcal{W}}, \mathcal{O}) \\
\boxed{H^1(Y'', \mathcal{O})} \arrow[u, phantom, sloped, "\cong"] \arrow[r, "res"] & \boxed{H^1(Y', \mathcal{O})} \arrow[u, phantom, sloped, "\cong"] \arrow[d] \\
H^1(Y_2, \mathcal{O}) \arrow[u] \arrow[r, "res"] \arrow[ur, phantom, "\circlearrowleft"] & H^1(Y_1, \mathcal{O})
\end{tikzcd}$$

Leray's Thm — Leray's Thm

(finite dim'l image by Key step)

Since $\operatorname{im} H^1(Y_2, \mathcal{O}) \subseteq \operatorname{im}\left(H^1(Y'', \mathcal{O}) \xrightarrow{res} H^1(Y', \mathcal{O})\right)$ & the later is finite-dimensional by the key step, this confirms the statement. $\square$

Corollary: If X is a compact R.S $\dim H^1(X, \mathcal{O}) < \infty$

Pf/ Take $Y = X$ in the Finiteness Thm (res = id).

§19.2 Consequences of Finiteness Theorem:

These finiteness results have several important consequences regarding construction of non-constant holomorphic / meromorphic functions on Riemann surfaces.

The Riemann-Roch Thm will be of the same spirit.

Corollary 1: Let X be a R.S. & $Y \Subset X$ a relative compact open subset of X. Then, given any $a \in Y$ $\exists$ $f \in \mathcal{M}(Y)$ non-constant holomorphic on $Y \setminus \{a\}$ with a pole at a. In particular, if X is compact, we can take $Y = X$.

Proof: Take a local coord chart (U, z) around a $\quad \begin{array}{ccc} U & \xrightarrow{\sim} & \mathbb{D} \\ a & \longmapsto & 0 \end{array}$ with $U \subset Y$

Take the open cover $\underline{\mathcal{U}} = \{ U_1 = U, \ U_2 = X \setminus \{a\} \}$ of X.

$U_{12} = U \setminus \{a\} \simeq \mathbb{D}^*$ There are no triple intersections

Claim: $H^1(\underline{\mathcal{U}}, \mathcal{O}) \xrightarrow{\text{res}} H^1(\underline{\mathcal{U}} \cap Y, \mathcal{O})$ has finite dim'l image.

Pf/

$$\begin{tikzcd}
H^1(\underline{\mathcal{U}}, \mathcal{O}) \arrow[r, "\text{res}"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & H^1(\underline{\mathcal{U}} \cap Y, \mathcal{O}) \arrow[d, hook] \\
\boxed{H^1(X, \mathcal{O}) \xrightarrow{\text{res}} H^1(Y, \mathcal{O})} &
\end{tikzcd}$$

finite dim'l image by Thm.

Write $k := \dim_{\mathbb{C}} \left(\text{res}(H^1(\underline{\mathcal{U}}, \mathcal{O})) \subseteq H^1(\underline{\mathcal{U}} \cap Y, \mathcal{O}) \right)$

Now: $\frac{1}{z^j} \in \mathcal{O}(U_1 \cap U_2)$ for all $j \geq 1$

Since $Z^1(\underline{\mathcal{U}}, \mathcal{O}) = \mathcal{O}(U_1 \cap U_2)$ (no triple intersections on $\underline{\mathcal{U}}$), we write $\boxed{\zeta_j} := [z^{-j}] \in H^1(\underline{\mathcal{U}}, \mathcal{O})$ for $j = 1, \dots, k+1$

- We have more than the dimension of $\text{res}(H^1(\underline{\mathcal{U}}, \mathcal{O})) \subseteq H^1(\underline{\mathcal{U}} \cap Y, \mathcal{O})$, so

  $\{ z^{-j}|_Y : j = 1, \dots, k+1 \}$ are linearly dependent modulo $B^1(\underline{\mathcal{U}} \cap Y, \mathcal{O})$

  Pick $c_1, \dots, c_{k+1} \in \mathbb{C}$, not all 0 with $\boxed{\sum_{j=1}^{k+1} c_j z^{-j} \Big|_{U_{12} \cap Y} = \partial f = f_2 - f_1}$ $(*)$

  for some $f \in C^0(\underline{\mathcal{U}} \cap Y, \mathcal{O})$

Here, $f = (f_1, f_2)$ with $f_1 \in \mathcal{O}(U_1 \cap Y) = \mathcal{O}(U)$

$f_2 \in \mathcal{O}(Y^*)$

Rearranging $(*)$ we get $f_2 = \underbrace{\sum_{j=1}^{k+1} c_j z^{-j}}_{\substack{a \text{ is a pole} \\ \text{of this function} \\ \text{on } U}} + \underbrace{f_1}_{\in \mathcal{O}(U)}$ on $U_{12} \cap Y = U^*$

Thus, $f_2 \in \mathcal{M}(Y)$ is non-constant, holomorphic on $Y \setminus \{a\}$ & has a pole at a □

Corollary 2: Fix a compact RS X & n distinct pts $\{a_1, \ldots, a_n\}$ on X. Pick $c_1, \ldots, c_n \in \mathbb{C}$. Then $\exists\, f \in \mathcal{M}(X)$ with $f(a_j) = c_j$ $\forall j = 1, \ldots, n$.

Proof: Pick $f_i \in \mathcal{M}(X)$ s.t f_i has a pole at a_i & it's holomorphic on $X \setminus \{a_i\}$ (We can do so by Corollary 2)

- Next, we massage f_i to find $g_{ij} \in \mathcal{M}(X)$ with no poles at $\{a_1, \ldots, a_n\}$ with $g_{ij}(a_i) = 1$, $g_{ij}(a_j) = 0$

Choose $\lambda_{ij} \in \mathbb{C}^*$ so that $-\lambda_{ij} + f_i(a_j) \neq f_i(a_k)$ $\forall k \neq i$

(we can do so since $\{f_i(a_k) - f_i(a_j) : k \neq i, j\}$ is finite.)

Claim: $g_{ij}(z) = \dfrac{f_i(z) - f_i(a_j)}{f_i(z) - f_i(a_j) + \lambda_{ij}}$ works!

Pf/ • g_{ij} is holomorphic on neighborhoods of all $a_1, \ldots, a_n$ since the denominators don't vanish at a_k for $k \neq i$.

- At a_i the singularity comes from $f_i(z)$, so they cancel out

$$g_{ij}(z) = \frac{1 - f_i(a_j)/f_i(z)}{1 - \frac{(f_i(a_j) - \lambda_{ij})}{f_i(z)}} \xrightarrow[z \to a_i]{} 1$$

- $g_{ij}(a_j) = \frac{0}{\lambda_{ij}} = 0$ & $g_{ij}(a_i) = 1$. □

To finish $h_i := \prod_{j \neq i} g_{ij} \in \mathcal{O}(X)$, holomorphic at $a_1, \dots, a_n$

& $h_i(a_k) = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{else} \end{cases}$

Then $f = \sum_{j=1}^{n} c_j h_j$ does the trick!

Corollary 3: Fix X non-compact RS & $Y \Subset X$ open. Then $\exists$ holomorphic function $f: Y \longrightarrow \mathbb{C}$ which is non-constant on each connected component of Y.

Proof We need to replace Y by a relatively compact domain containing it.

- **Claim 1:** Build Y_1 open with $Y \Subset Y_1 \Subset X$ & Y_1 connected

Pf/ For each $a \in Y$, pick a local chart $U \xrightarrow{\varphi_a} \mathbb{D}$ around $a \in Y$

Then: $\varphi_a^{-1}(D_{\frac{1}{4}}(0))$ covers Y.

Then $\overline{Y} \subseteq \bigcup_{i \in I} \varphi_{a_i}^{-1}(\overline{D}_{\frac{1}{4}}(0)) \subseteq \bigcup_{i \in I} \varphi_{a_i}^{-1}(D_{\frac{1}{2}}(0))$

Since $\overline{Y}$ is compact, we can find a finite subcover using $a_{i_1} \dots a_{i_n}$

To build Y_1, we need to connect $U_j := \varphi_{a_{i_j}}^{-1}(\overline{D}_{\frac{1}{2}}(0))$ $j = 1, \dots, n$

Since X is pathwise connected, for each $j = 2, \dots, n$ we can find a path $\gamma_j : [0,1] \longrightarrow X$ starting at a_{i_1} & ending at a_{i_j}.

We can find a connected open W_j containing $\operatorname{im} \gamma_j$ (use a cover of $\operatorname{im} \gamma_j$ by finitely many local charts & use the fact that $\gamma_j[0,1]$ is connected to conclude that the union of these charts is connected).

Then $Y_1 = \bigcup_{j=1}^{n} U_j \cup \bigcup_{j=2}^{n} W_j$ is open, connected & $\overline{Y} \subseteq \bigcup_{j=1}^{n} U_j \subset Y_1$ $\square$

- Claim 2: $Y_1 \setminus Y \neq \emptyset$

Pf/ Argue by contradiction. If $\overline{Y} \subset Y_1 = Y$, then Y is both open & closed. Since $Y \neq \emptyset$ & X is connected, we have $X = Y$ so $X = \overline{Y}$ is compact. This is a contradiction. $\square$

- Pick $a \in Y_1 \setminus Y$ & use Corollary 1 applied to $Y_1 \Subset X$ to find a meromorphic function f on Y_1 non-constant with a pole at Y & holomorphic outside a.

In particular, since Y_1 is connected & each connected component of Y is open, f cannot be constant on them, otherwise it would be constant on Y_1. This cannot happen $\square$

Before stating & proving our last corollary, we need the following lemma

Lemma: Fix X a R.S. then $H^1(X, \mathcal{O})$ is a module over $\mathcal{O}(X)$.

Proof Multiplication by $f \in \mathcal{O}(X)$ defines a linear endomorphism

$$\begin{array}{ccc} H^1(\underline{\mathcal{U}}, \mathcal{O}) & \longrightarrow & H^1(\underline{\mathcal{U}}, \mathcal{O}) \\ [(g)] & \longmapsto & [(fg)] = [(f|_{U_{ij}}\, g_{ij})_{ij}] \end{array}$$

for any open covering $\underline{\mathcal{U}} = (U_{ij})_{ij}$ of X, compatible with refinement maps $\tau^{\underline{\mathcal{U}}}_{\underline{\mathcal{V}}}$, and thus yields a ! linear map $H^1(X, \mathcal{O}) \longrightarrow H^1(X, \mathcal{O})$.

The required axioms for modules $/ \mathcal{O}(X)$ are easy to check. $\square$

Corollary 4: Assume X is a non-compact RS & pick Y, Y' open, relatively compact subsets of X with $Y \Subset Y' \Subset X$. Then

$$\operatorname{Im}\left(H^1(Y', \mathcal{O}) \xrightarrow{\text{res}} H^1(Y, \mathcal{O}) \right) = \{0\}$$

Proof: We know by Theorem §19.1, that the image is finite-dimensional $/\mathbb{C}$.
Say its value is n & pick $\zeta_1, \ldots, \zeta_n \in H^1(Y', \mathcal{O})$ with

$$L = \operatorname{span}(\operatorname{res}(\zeta_1), \ldots, \operatorname{res}(\zeta_n)) = \operatorname{Im}(\operatorname{res})$$

By Corollary 3 applied to $Y' \Subset X$ $\exists\, f \in \mathcal{O}(Y')$ non-constant on any connected component of Y'. Multiplying by f gives an element in $\operatorname{End}(H^1(Y', \mathcal{O}))$ by Lemma §19.2.

$$\Rightarrow \underbrace{f_{|Y} \zeta_{i|Y}}_{\in L} = \sum_{j=1}^{n} c_{ij} \zeta_{j|Y} \quad \text{on } Y \quad \text{with} \quad c_{ij} \in \mathbb{C}$$

Next, we define $F := \det(f\,\mathrm{Id} - (c_{ij})) \in \mathcal{O}(Y')$

Claim 1: F is not identically 0 on any connected component of Y'.
Pf/ $F = G(f)$ where $G := \det(T\,\mathrm{Id} - (c_{ij})) \in \mathbb{C}[T]$ monic of degree n.
Write Z for a connected component of Y'. If $F_{|Z} \equiv 0$ on Z, then $\operatorname{im}(f_{|Z})$ lies in the set of n roots of G & it is connected (f is continuous on Y'), so $f_{|Z}$ is constant. This contradicts our assumption on f. $\square$

Claim 2: $F_{|Y} \cdot \zeta_i = 0$ on $H^1(Y, \mathcal{O})$ $\Longrightarrow F_{|Y} \cdot L = 0$

Pf/ • Let's see it first for $n=2$. Write
$$f\zeta_1 = a\zeta_1 + b\zeta_2$$
$$f\zeta_2 = c\zeta_1 + d\zeta_2$$

$$\Rightarrow F := \det\begin{pmatrix} f-a & -b \\ -c & f-d \end{pmatrix} = f^2 - (a+d)f + (ad-bc)$$

$$F_{|Y} \cdot \zeta_1 = f^2 \cdot \zeta_1 - (a+d) f \cdot \zeta_1 + (ad-bc) \cdot \zeta_1$$

$$= f(\underbrace{f \zeta_1}_{= a\zeta_1 + b\zeta_2}) - (a+d)(\underbrace{f \cdot \zeta_1}_{= a\zeta_1 + b\zeta_2}) + (ad-bc) \cdot \zeta_1$$

$$= a(a\zeta_1 + b\zeta_2) + b \cdot (c\zeta_1 + d\zeta_2) - (a+d)(a\zeta_1 + b\zeta_2) + (ad-bc)\zeta_1$$

$$= (a^2 + bc - a^2 - ad + ad - bc)\zeta_1 + (ab + bd - ab - db)\zeta_2 = 0 \checkmark$$

$F_{|Y} \cdot \zeta_2 = 0$ by symmetry.

- General argument : $A = (f\,\mathrm{Id} - (c_{ij}))$ satisfies $\mathrm{adj}(A) \cdot A = F \cdot \mathrm{Id}$

Since $A_{|Y} \cdot \zeta_{i|Y} = 0$ on Y $\forall i$ $\Rightarrow$ $F_{|Y} \cdot \zeta_{i|Y} = 0$ $\forall i$.
↳ basis for L.

$\square$

By Claim 1, the zeroes of $F \in \mathcal{O}(Y')$ are discrete. Thus, we can pick a Leray covering $\underline{\mathcal{U}} = (U_i)_{i \in I}$ of Y' for $\mathcal{O}_{Y'}$ s.t

① each U_i has at most one zero of F

② F has no zero on U_{ij}

(Recall: $H^1(\Delta_R(0), \mathcal{O}) = 0$ $\forall R$. Thus, we can pick local charts around 0's containing exactly one zero & chart around the remaining pts containing no zeroes of F. This collection will be a Leray covering for $\mathcal{O}$)

- In particular, $F_{|U_{ij}} \in \mathcal{O}^*(U_i \cap U_j)$ by ②
- We have $H^1(Y', \mathcal{O}) \simeq H^1(\mathcal{U}, \mathcal{O})$ by Leray's Theorem (§17.2)

Claim 3: $\mathrm{res} \equiv 0$.

Pf/ Pick $h \in H^1(Y', \mathcal{O})$ & it's representative $(h_{ij})_{ij} \in Z^1(\underline{\mathcal{U}}, \mathcal{O})$.

It follows that we can find $(g_{ij})_{ij} \in Z^1(\underline{U}, \mathcal{O})$ with $h_{ij} = F_{|U_{ij}} \cdot g_{ij}$

($F_{|U_{ij}} \in \mathcal{O}^*(U_i \cap U_j)$ & $F \in \mathcal{O}(Y')$ so the cocycle equation for $g_{ij} = \frac{1}{F_{|U_{ij}}} h_{ij}$ is inherited from that of F).

Fix $g \in H^1(Y', \mathcal{O})$ to be the class associated to $(g_{ij})_{ij}$

Then $h = F \cdot g$. Thus, $\mathrm{res}(h) = h_{|Y} = F_{|Y} \cdot \underbrace{g_{|Y}}_{\in L} \overset{\text{by Claim 2}}{=} 0$.

Conclusion: $\mathrm{res}(h) = h_{|Y} = 0$ as we wanted