Lecture XX: Divisors on RS

Last Time: Finiteness Thm & its corollaries

THM: X RS, Y_1, Y_2 rel. compact open subsets $Y_1 \Subset Y_2 \Subset X$. Then, the restriction map $H^1(Y_2, \mathcal{O}) \xrightarrow{\text{res}} H^1(Y_1, \mathcal{O})$ has finite dimensional image.

In particular, if X is compact, then genus$(X) = \dim_{\mathbb{C}} H^1(X, \mathcal{O}) < \infty$.

Consequences:

① X any RS, $Y \Subset X$ & $a \in Y$. Then $\exists\, f \in \mathcal{M}(Y)$ non-constant, holo on $Y \setminus \{a\}$ with a pole at a

② X compact RS, $\{a_1, \ldots, a_n\}$ n pts on X & $\{c_1, \ldots, c_n\} \subseteq \mathbb{C}$. Then $\exists\, f \in \mathcal{M}(X)$ with $f(a_j) = c_j$ $\forall j = 1, \ldots, n$

③ X non-compact RS, $Y \Subset X$. Then $\exists\, f \in \mathcal{O}(Y)$ non-constant on each connected component of Y

④ X non-compact RS, $Y \Subset Y' \Subset X$. Then $\text{res}: H^1(Y', \mathcal{O}) \longrightarrow H^1(Y, \mathcal{O}) \equiv 0$

(Compare with THM: X was general in THM & now it's non compact. We used ③ in the proof)

Lemma: Fix X a R.S. then $H^1(X, \mathcal{O})$ is a module over $\mathcal{O}(X)$.

Proof Multiplication by $f \in \mathcal{O}(X)$ defines a linear endomorphism

$$H^1(\mathfrak{U}, \mathcal{O}) \longrightarrow H^1(\mathfrak{U}, \mathcal{O})$$
$$[(g)] \longmapsto [(fg)] = [(f|_{U_{ij}}\, g_{ij})_{ij}]$$

compatible with refinements $\mathfrak{V} < \mathfrak{U}$

NEXT: How many lin indep mero functions on compact RS can we have with certain (order) restrictions on poles?

§20.1 Divisors on R.S.

Definition: A divisor D on a RS X is a map $D: X \to \mathbb{Z}$ where for each $K \subset X$ compact we have $\text{Supp}(D|_K) := \{x \in K : D(x) \neq 0\}$ is finite.

. $\mathrm{Div}(X) = \{\text{divisors on } X\}$ is an ab. group under $+$

Obs: $\mathrm{Div}(X)$ is a poset: $D \leq D'$ if $D(x) \leq D'(x) \ \forall x \in X$

Q: How to build divisors?

Examples: 1) Divisors from meromorphic functions on a R.S. X (= principal divisors)

Fix $Y \subset X$ open & $a \in Y$

Define: $\mathrm{ord}_a : \mathcal{M}(Y) \setminus \{0\} \longrightarrow \mathbb{Z} \cup \{\infty\}$, $f \longmapsto \mathrm{ord}_a(f)$

$$\mathrm{ord}_a(f) = \begin{cases} 0 & \text{if } f \text{ holo at } a \text{ \& } f(a) \neq 0 \\ k & \text{if } a \text{ is a zero of } f \text{ of order } k \\ -k & \text{if } a \text{ is a pole of } f \text{ of order } k \\ \infty & \text{if } f \equiv 0 \text{ in a nbhd of } a \end{cases}$$

$\Rightarrow$ Given $f \in \mathcal{M}(X) \setminus \{0\}$ we have a divisor on X (a ppal divisor)

$$D = (f) : X \longrightarrow \mathbb{Z} \qquad a \longmapsto \mathrm{ord}_a(f)$$

Obs: Zeros & Poles of X are discrete, so $|\mathrm{Supp}(f)|_K| < \infty \quad \forall K \subset X$ compact.

Note . $(f) \geq 0 \iff f \in \mathcal{O}(X) \setminus \{0\}$.

. Principal divisors form a subgroup of $\mathrm{Div}(X)$ since $(fg) = (f) + (g)$.
$(1/f) = -(f)$

Def: $D, D' \in \mathrm{Div}(X)$ are equivalent (linearly) if $D - D' = (f)$ for some $f \in \mathcal{M}(X) \setminus \{0\}$

Write $\mathrm{Pic}(X) := \mathrm{Div}(X)/\sim$

Ex: $D : \mathbb{C} \longrightarrow \mathbb{Z}$ $\quad D(0) = 3, \ D(x) = 0 \ \forall x \neq 3 \quad \Rightarrow D = (z^3)$

$D : \mathbb{P}^1 \longrightarrow \mathbb{Z}$ $\quad D(0) = 3, \ D(x) = 0 \ \forall x \neq 3$ is not principal! $\mathcal{O}(\mathbb{P}^1) = \mathbb{C}$.

Q!: How to determine linearly equivalence?

2) Divisors from meromorphic 1-forms on a RS X. (= canonical divisors)

. **Def** $\mathcal{M}^{(1)}$: sheaf of meromorphic 1-forms on X

On a chart (U, z), $\omega \in \mathcal{M}^{(1)}(U)$ if $\omega = f\,dz$ for $f \in \mathcal{M}(U)$

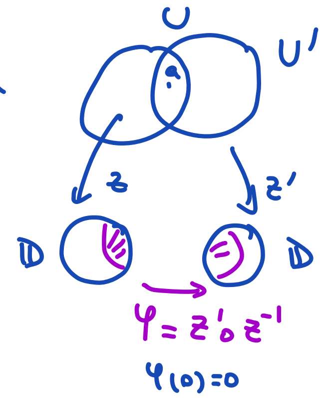
Glueing works as we did for holomorphic 1-forms.

- Given $Y \subset X$ open, $a \in X$ & $\omega \in \mathcal{M}^{(1)}(Y)$, use a local chart (U,z) around a to express $\omega|_U = f\,dz$ for $f \in \mathcal{M}(U)$.

Def: $\operatorname{ord}_a(\omega) = \operatorname{ord}_0(f)$

Lemma 1: $\operatorname{ord}_a(\omega)$ is well defined

Pf / If we have $U \underset{z}{\simeq} \mathbb{D} \underset{z'}{\simeq} U'$ two charts around a



$\omega = \underset{(z)}{f}\,dz = \underset{(z')}{g}\,dz'$ is given by the glueing condition:

$$g\,dz' = g(z' \circ z^{-1} \circ z)\, d(z' \circ z^{-1} \circ z)$$
$$= g(\varphi \circ z)\, d(\varphi \circ z) = \underbrace{(g \circ \varphi)_{(z)}\, \varphi'}_{= f(z)}\, dz$$

$$\operatorname{ord}_0(f) = \operatorname{ord}_0(g \circ \varphi) + \underbrace{\operatorname{ord}_0 \varphi'}_{=0} \underset{\uparrow}{=} \operatorname{ord}_{\varphi(0)}(g) = \operatorname{ord}_0(g)$$
(φ is invertible at 0)

$\Rightarrow$ Given $\omega \in \mathcal{M}^{(1)}(X) \setminus \{0\}$ we define a new divisor (canonical div)

Definition:
$$D = (\omega) : X \longrightarrow \mathbb{Z}$$
$$a \longmapsto \operatorname{ord}_a(\omega)$$

Obs: Given $K \subseteq X$ compact, cover it with finitely many charts $\varphi^{-1}(\mathbb{D}_{\frac{1}{2}}(0))$ coming from (U,z) $U \overset{\varphi}{\simeq} \mathbb{D}$.

write $\omega|_U = f\,dz$ $f \in \mathcal{M}(U)$ $\Rightarrow$ f has finitely many zeros & poles on $\varphi^{-1}(\overline{\mathbb{D}}_{\frac{1}{2}}(0))$, which cover K.

Conclude $|\operatorname{Supp}(\omega)|_K| < \infty$, so (ω) is a divisor.

Lemma 2: Canonical divisors are linearly equivalent

Pf/ $(f\omega) = (f) + (\omega)$ for $\omega \in \mathcal{M}^{(1)}(X)$, $f \in \mathcal{M}(X)$.

Given $\omega_1, \omega_2 \in \mathcal{M}^{(1)}(X)$ locally on charts (U,z) we can write

$\omega_1|_U = f_1\,dz$ $\omega_2|_U = f_2\,dz$ $\Rightarrow$ $g_U = \frac{f_1}{f_2} \in \mathcal{M}(U)$ & $g_U \cdot \omega_2|_U = \omega_1|_U$

Note g_U is unique with this property. Since ω_1, ω_2 are defined on X, the sections g_U agree on overlaps. Since $\mathcal{O}$ is a sheaf, we get a $! \, g \in \mathcal{O}(X)$ with $g|_U = g_U$ $\forall (U,z)$ local chart.

<u>Conclude:</u> $(\omega_1) = (g\,\omega_2) = (g) + (\omega_2)$ so $(\omega_1) \sim (\omega_2)$

<u>Ex.</u>: $D : \mathbb{C} \longrightarrow \mathbb{Z}$ $\quad D = 3(0) = (z^3)$ is canonical $= (z^3 dz)$

$D : \mathbb{P}^1 \longrightarrow \mathbb{Z}$ $\quad D = 3\cdot(0)$ is not canonical

If $D = (\eta)$ then $\eta = f_{(z)} dz = (z^3 + \text{hot})\, dz$ on U_0.

Behavior at ∞? It's at worst a pole, so f is a polynomial!

But then $z = \frac{1}{w}$ on $U_0 \cap U_\infty$

$$\Rightarrow f_{(z)}\, dz = f\left(\tfrac{1}{w}\right) \frac{-1}{w^2}\, dw = g_{(w)}\, dw$$

$$\Rightarrow \operatorname{ord}_\infty(g) = \deg f + 2$$

So $(\eta) = 3[0] - (\deg f + 2)[\infty] \neq 3[0]$.

Q2: How to decide if a divisor is canonical?

<u>§20.2 Degree of divisors</u>

From now on, we'll restrict to the compact case.

Fix X = compact RS. & $D \in \operatorname{Div}(X)$. $\Rightarrow$ Write $D = \sum_{i=1}^{N} D_{(x_i)} x_i$

$(\{x \in X : D_{(x)} \neq 0\} = \{x_1, \dots, x_N\})$

<u>Definition</u>: The <u>degree</u> of a divisor D on a compact RS X is defined as

$$\deg(D) = \sum_{x \in X} D_{(x)}$$

(the sum involves finitely many non-zero terms!)

$\rightsquigarrow$ Degree map $\deg : \operatorname{Div}(X) \longrightarrow \mathbb{Z}$

Lemma: deg is a group homomorphism

Ex $D = 3[0]$ on $\mathbb{P}^1$ $\deg(D) = 3$

$D = 3[0] - 4\cdot[\infty]$ on $\mathbb{P}^1$ $\deg(D) = -1$

Proposition: $\deg(f) = 0 \quad \forall f \in \mathcal{M}(X) \setminus \{0\}$.

⚠ $f : \mathbb{P}^1 \longrightarrow \mathbb{C}$ meromorphic is a proper holo map $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$

If f is not constant, it has a degree $= |f^{-1}(y)| \quad \forall y \in \mathbb{P}^1$. (Theorem §5.3)

This is NOT the degree of the divisor (f).

Proof: • If f is constant, then $(f) = 0$ so $\deg(f) = 0$

• If f is not constant, the size of each fiber of f is the same (= degree of f)

Corollary 2 §5.3 says # zeroes of f = # poles of f (counted with mult)

$$\Rightarrow \sum_{a \text{ zero}} \operatorname{ord}_a(f) = \sum_{b \text{ pole}} -\operatorname{ord}_b(f) \quad \& \text{ so } \deg(f) = 0 .$$

$$0 = \sum_{a \text{ zero}} \operatorname{ord}_a(f) + \sum_{b \text{ pole}} \operatorname{ord}_b(f) = \deg(f) . \qquad \square$$

Remark: This gives a necessary condition (not sufficient!) to be a principal divisor on a compact RS.

Observation: We'll see later that if K_X is a canonical divisor on a compact RS X, then $g = \text{genus}(X) = \frac{1}{2}\deg K + 1$. (necessary criterion for being a canonical divisor)

Equivalently: $\deg K = 2g - 2$.

Q: Can we find $\omega \in \mathcal{M}^{(1)}(X)$ that is holomorphic & nowhere vanishing on X?

If so $K = (\omega) = 0$ $(\operatorname{ord}_x(\omega) = 0 \quad \forall x \in X)$.

Then $\deg K = 0$ ie $g = 1$ so $X = E = \mathbb{C}/\Lambda$ (elliptic curve!)

Can we find such ω on E?

A = YES! $dz \in \mathcal{E}'(\mathbb{C})$ gives a holomorphic 1-form on E (it's invariant under translations by the lattice)

$\Rightarrow K = (dz)$ is a canonical divisor on $E = \mathbb{C}/\Lambda$.

<u>Examples on $\mathbb{P}^1$</u>: Write $\mathbb{P}^1 = U_0 \cup U_\infty$.

① dz on U_0 is holomorphic & nowhere vanishing but has a pole of order 2 at ∞.

<u>Why?</u> $z = \frac{1}{w}$ on $U_0 \cap U_\infty \Rightarrow dz = \frac{-1}{w^2}\,dw$ ⇝ so we see a pole of order 2

$\Rightarrow K = -2[\infty]$ $\quad \deg K = -2 \; (= 2 \cdot 0 - 2)$ ✓

③ Another expression for a canonical divisor on $\mathbb{P}^1$:

$\eta = \frac{dz}{z} \in \mathcal{M}^{(1)}(\mathbb{P}^1)$ $\qquad \frac{dz}{z} = w\, d\left(\frac{1}{w}\right) = w \frac{-1}{w^2}\, dw = -\frac{dw}{w}$

$\Rightarrow \eta$ has poles at 0 & ∞, both of order 1.

$\Rightarrow K = (\eta) = -[0] - [\infty]$ $\quad \deg K = -2$ ✓

<u>Clearly</u> $(dz) \sim \left(\frac{dz}{z}\right)$ (as we expected from Lemma 2 §20.1)

§ 20.3 Sheaf $\mathcal{O}_D$

Fix X any RS & $D \in \mathrm{Div}(X)$

<u>Definition</u>: $\mathcal{O}_D$ a sheaf on X

(1) $U \subset X$ open $\quad \mathcal{O}_D(U) = \{ f \in \mathcal{M}(U) : \mathrm{ord}_x(f) \geq -D(x) \;\; \forall x \in U \}$ $\qquad (f) + D|_U \geq 0$

(2) restriction maps inherited from $\mathcal{M}$

neg wts of D — give $\geq$ order of zeros of f

pos — give worst order of poles of f

• Gluing condition on $\mathcal{M}$ is compatible with $(f) + D|_U \geq 0 \Rightarrow \mathcal{O}_D$ is a sheaf

<u>Q</u> How to think about $\mathcal{O}_D(U)$?

<u>A:</u> Assume Supp D is finite (it's discrete) & write

$$D = -\sum_{i=1}^{N} a_i[p_i] + \sum_{j=1}^{M} b_j[p_j] \qquad \text{with } a_i, b_j > 0 \ \forall i,j$$

(we allow N or $M = 0$)

$\Rightarrow \mathcal{O}_D(U) = \{ f \in \mathcal{M}(U) : \operatorname{ord}_x(f) \geq -D_{(x)} \ \forall x \in U \}$

- For $x \notin \operatorname{Supp}(D)$ $\quad \operatorname{ord}_x(f) \geq 0$, so f is holomorphic at x
- For $x = p_i$ $\quad \operatorname{ord}_{p_i}(f) \geq +a_i$, so p_i is a zero of f of order $\geq a_i$

$$\Rightarrow f_{(z)} = (z-p_i)^{a_i} \varphi \qquad \varphi \in \mathcal{O} \text{ near } p_i$$

- For $x = q_j$ $\quad \operatorname{ord}_{q_j}(f) \geq -b_j$ so at worst, q_j is a pole of f of order $\leq b_j$.

$$\Rightarrow f(z) = \frac{\varphi}{(z-q_j)^{b_j}} \qquad \varphi \in \mathcal{O} \text{ near } q_j$$

In short: $\mathcal{O}_D$ prescribes behavior of zeros & poles for meromorphic functions on U.

Q: When is $\mathcal{O}_D(U) \neq \{0\}$? $\quad$ ($0 \in \mathcal{O}_D(U)$ $\forall U$ open since $\operatorname{ord}_x(0) = \infty$ $\forall x$)

A: Look at stalks & work with charts!

$(\mathcal{O}_D)_p = \{ f \in \mathcal{M}_p \ \ \operatorname{ord}_p(f) \geq -D_{(p)} \}$ (prescribed behavior at p & $D_{(x)} = 0$ $\forall x \in \overline{U} \setminus \{p\}$ $\ \overline{U} \simeq \overline{\mathbb{D}}_{\frac{1}{2}}(0)$)

$\Rightarrow$ Laurent series exp near p is $\quad f = \dfrac{1}{(z-p)^{D(p)}} \varphi \quad \varphi \in \mathbb{C}[\![z-p]\!]$.

So for each p $\exists V \ni p$ open with $\mathcal{O}_D(V) \neq \{0\}$.

Ex: $\mathcal{O}_0 = \mathcal{O}$

On $\mathbb{P}^1$: $\mathcal{O}_{3[0]} = ?$ $\qquad$ Look at charts (U, z). $\quad U \simeq \mathbb{D}$

$$\mathcal{O}_{3[0]}(U) = \begin{cases} \mathcal{O}(U) & \text{if } 0 \notin U \\ \mathcal{O}(U)\left(\frac{1}{z^3}\right) & \text{if } 0 \in U \end{cases}$$

$\Rightarrow$ Stalks: $(\mathcal{O}_{3[0]})_p = \begin{cases} \mathbb{C}[\![z-p]\!] & \text{if } p \neq 0 \\ \left(\frac{1}{z^3}\right)\mathbb{C}[\![z]\!] & \text{if } p = 0 \end{cases}$

Proposition: If $D \sim D'$, then $\mathcal{O}_D \simeq \mathcal{O}_{D'}$

Proof: Write $D = (g) + D'$ for $g \in \mathcal{M}(X) \setminus \{0\}$

$$\Rightarrow \mathcal{O}_D(U) = \{ f \in \mathcal{M}(U) : \mathrm{ord}_x(f) \geq -D(x) = -\mathrm{ord}_x(g|_U) - D'(x) \ \forall x \in U \}$$
$$= \{ f \in \mathcal{M}(U) : \mathrm{ord}_x(fg) \geq -D'(x) \ \forall x \in U \}$$
$$= \{ f \in \mathcal{M}(X) : fg \in \mathcal{O}_{D'}(U) \}$$

$$\Rightarrow \begin{array}{ccc} \mathcal{O}_D(U) & \xrightarrow{\cdot g|_U} & \mathcal{O}_{D'}(U) \\ f & \longmapsto & fg|_U \end{array} \quad \forall U.$$

invertible because f, g are meromorphic

$\Rightarrow \mathcal{O}_D \xrightarrow{\cdot g} \mathcal{O}_{D'}$ is the isomorphism.

Summary: If X is a compact RS, we have $\dim H^1(X, \mathcal{O}) < \infty$

Q: What happens to $H^1(X, \mathcal{O}_D)$ for $D \in \mathrm{Div}(X)$?

A $\dim_{\mathbb{C}} H^1(X, \mathcal{O}_D) < \infty$! (This is part of Riemann-Roch.)

- How do we compute this dimension? Eg. If we have a Leray covering of X for $\mathcal{O}_D$, then task is easier. Can we find such a Leray covering?

  Hint: $H^1(\mathbb{D}, \underset{= \mathcal{O}_0}{\mathcal{O}}) = 0$. by Theorem §17.3, consequence of Dolbeault's Thm.

Theorem: Fix X a compact R.S, $D \in \mathrm{Div}(X)$ & $\underline{\mathcal{U}} = (U_i)_{i \in I}$ an open covering by local charts $U_i \simeq \mathbb{D}$ $\forall i$. Then, $\underline{\mathcal{U}}$ is a Leray covering for $\mathcal{O}_D$, ie $H^1(U_i, \mathcal{O}_D) = 0$ $\forall i$.

Pf/ Next Time!