Lecture XXI : The Riemann-Roch Theorem..

Recall: Divisors on RS $=$ Div(X) ab gp

- $D: X \to \mathbb{Z}$ $\quad |\operatorname{supp} D_{|K}| < \infty$ $\quad \forall K \subset X$ compact

Examples: ① Ppal div : $D = (f)$ $\quad f \in \mathcal{M}(X) \setminus \{0\}$

② Canonical div : $D = (\omega)$ $\quad \omega \in \mathcal{M}^{(1)}(X) \setminus \{0\}$ $\quad \omega_{|U} = f\,dz$ on (U,z), $f \in \mathcal{M}(U)$

- If X = compact RS, we have $\deg(D) = \sum_{x \in X} D(x) \in \mathbb{Z}$ $\rightsquigarrow$ $\deg : \operatorname{Div}(X) \to \mathbb{Z}$ gp homomorphism

Lemma : $\deg(f) = 0$ $\quad \forall f \in \mathcal{M}(X) \setminus \{0\}$.

- $\mathcal{O}_D$: sheaf on X with $\mathcal{O}_D(U) = \{ f \in \mathcal{M}(U) : \operatorname{ord}_x(f) \geq -D(x) \ \forall x \in U \}$ $\forall U$ open

($\mathcal{O}(D)$ is Algebraic Geometry notation)

Note: $0 \in \mathcal{O}_D(U)$ $\forall U$ $\quad (\operatorname{ord}_x(0) = \infty)$

Stalks : $(\mathcal{O}_D)_p = (z-p)^{-D(p)} \cdot \mathbb{C}\{z-p\}$

$\hookrightarrow$ convergent power series in $(z-p)$.

Example : $\mathcal{O}_0 = \mathcal{O}$

Proposition : If $D = D' + (g)$ for $g \in \mathcal{M}(U) \setminus \{0\}$, then $\mathcal{O}_D \xrightarrow[\sim]{\cdot g} \mathcal{O}_{D'}$

§ 21.1 More on $\mathcal{O}_D$:

Recall $H^1(\mathbb{D}, \mathcal{O}) = 0$, so local charts in X with $U_i \simeq \mathbb{D}$ are Leray for $\mathcal{O}$. (of order 1)

Same is true for $\mathcal{O}_D$ if X is compact!

Theorem: Fix X a compact RS, $D \in \operatorname{Div}(X)$ & $\underline{\mathcal{U}} = (U_i)_{i \in I}$ an open covering by local charts with $U_i \simeq \mathbb{D}$ $\forall i$. Then, $\underline{\mathcal{U}}$ is a Leray covering of order 1 for $\mathcal{O}_D$, ie $H^1(U_i, \mathcal{O}_D) = 0$ $\forall i$.

Proof: $\mathcal{O}_D(U_i) = \{ f \in \mathcal{M}(U_i) : \operatorname{ord}_x(f) \geq -D(x) \ \forall x \in U_i \}$

Since X is compact, Supp D is finite. The same is true for $D_{|U_i}$

- If $\operatorname{Supp} D|_{U_i} = \phi$, then $\mathcal{O}_D(V) = \mathcal{O}(V)$ $\forall V \subseteq U$, so
$H^1(U_i, \mathcal{O}_D) = H^1(U_i, \mathcal{O}) = 0$ by Theorem §17.3, consequence of Dolbeault's Thm

- Next, assume $\operatorname{Supp} D|_{U_i} \neq \phi$ & write:

$$D|_{U_i} = -\sum_{k=1}^{N} a_k [p_k] + \sum_{k=1}^{M} b_k [q_k] \qquad a_k, b_k > 0.$$

(we can have either N or $M = 0$)

$\Rightarrow f \in \mathcal{O}_D(U_i)$ is of the form $f_{(z)} = g_{(z)} \cdot h_{(z)}$ with $g \in \mathcal{O}(U_i)$
& $h \in \mathcal{M}(U_i)$ with prescribed 0's & poles on $q_1, \dots, q_M$, $p_1, \dots, p_N$
(order $b_1 \dots b_M$ $a_1 \dots a_N$)

$$h_{(z)} = \prod_{k=1}^{N} (z - p_k)^{a_k} \Big/ \prod_{k=1}^{M} (z - q_k)^{b_k}$$

Given $\underline{\mathcal{V}}$ covering of $U_i \simeq \mathbb{D}$, want to show $H^1(\underline{\mathcal{V}}, \mathcal{O}_D) = 0$.

<u>Claim</u>: We pick a suitable refinement $\underline{\mathcal{W}} = (W_j)_{j \in J}$ where

① Each W_j has at most one element from $\{p_1, \dots p_N, q_1, \dots, q_M\}$

② each $W_{j\ell}$ has no p_s nor q_k.

(We can do this because our set is finite!)

<u>How?</u> Pick balls $D_{r_j}(p_j)$, $D_{s_\ell}(q_\ell)$ with $r_j < \operatorname{dist}(p_j, \partial U_i)/2$ $s_\ell < \operatorname{dist}(q_\ell, \partial U_i)/2$
pairwise disjoint, each ball in some V_{k_j}, V_{k_ℓ}.

Take $\underline{\mathcal{W}} := \{\underbrace{D_{\frac{r_j}{2}}(p_j)}_{= W_j}\}_j \cup \{\underbrace{D_{\frac{s_\ell}{2}}(q_\ell)}_{= W_{N+\ell}}\}_\ell \cup \{\underbrace{V_k \setminus \bigcup_{j,\ell} \left(\overline{D_{\frac{r_j}{2}}(p_j)} \cup \overline{D_{\frac{s_\ell}{2}}(q_\ell)} \right)}_{=: W_k}\}_k$

$$\tau : K \sqcup \{1, \dots, N, N+1, \dots, N+M\} \longrightarrow K \qquad \begin{aligned} j &\longmapsto k_j \\ \ell + N &\longmapsto k_{\ell - N} \\ k &\longmapsto k \end{aligned}$$

Then, $\underline{\mathcal{W}}$ covers U_i & $\underline{\mathcal{W}} < \mathcal{V}$.

$\square$

$\Rightarrow \mathcal{O}_D(W_{jk}) = \mathcal{O}(W_{jk})$ for all j,k $\quad (D|_{W_{jk}} = 0)$

Conclusion: $\quad Z^1(\underline{W}, \mathcal{O}_D) = Z^1(\underline{W}, \mathcal{O}) = B^1(\underline{W}, \mathcal{O})$

$$[\,H^1(\underline{W}, \mathcal{O}) \hookrightarrow H^1(\mathbb{D}, \mathcal{O}) = 0\,]$$

$\Rightarrow \xi \in Z^1(\underline{W}, \mathcal{O}_D)$ becomes $\quad \xi_{i_0 i_1} = \partial g = g_{i_1} - g_{i_0}$ on $U_{i_0 i_1}$

for $g_j \in \mathcal{O}(W_j)$

Need to adjust g_j to make it attain prescribed 0's & poles at q's & p's

Easiest: $\quad f = g - h$ for a suitable $h \in \mathcal{O}(U_i)$

What do we need? $\quad C^0(\underline{W}, \mathcal{O}_D) = (f_K)$ means

1. $f_K \in \mathcal{O}(W_K)$ for $k \in K$ $\qquad$ ok (is h is holo)

2. $f_j \in \mathcal{O}_D(D_{r_j}(p_j)) \rightsquigarrow f_j = \varphi_j (z-p_j)^{a_j}, \quad \varphi_j \in \mathcal{O}(D_{r_j}(p_j))$ ←

3. $f_\ell \in \mathcal{O}_D(D_{s_\ell}(q_\ell)) \rightsquigarrow f_\ell = \dfrac{\varphi_\ell}{(z-q_\ell)^{b_\ell}} \qquad \varphi_\ell \in \mathcal{O}(D_{s_\ell}(q_\ell))$ $\qquad$ ok because h is holo

If we use h holo, we are good on 1 & 3. For 2 we need to fix the order of a_k as a zero of g_ℓ.

- Relabel so that $g_1, \dots, g_n$ correspond to discs for $p_1, \dots, p_n$.

Then write $\quad g_K^{poly} = \sum_{j=0}^{a_K - 1} c_{Kj} (z-p_K)^j \qquad$ where $g_K = \sum_{j=0}^{\infty} c_{Kj} (z-p_K)^j$

in a nbhd of p_K in W_K

Our h should be a polynomial with

(*) $\quad h|_{z=a_j} = g_j^{poly} + (\text{hot}) \qquad \forall j.$

Our next Lemma confirms we can find such h.

$\Rightarrow$ Using $f_j = g_j - h$, we get $(f_j) \in C^0(\underline{W}, \mathcal{O}_D)$ & $\partial f = \xi$.

Conclude: $Z^1(\underline{W}, \mathcal{O}_D) = B^1(\underline{W}, \mathcal{O}_D)$ ie $H^1(\underline{W}, \mathcal{O}_D) = 0$.

$\Longrightarrow H^1(\underline{V}, \mathcal{O}_D) = 0$ since $\underline{W} < \underline{V}$.

Then $0 = \varinjlim_{\underline{V}} H^1(\underline{V}, \mathcal{O}_D) = H^1(U_i, \mathcal{O}_D) \quad \forall i$ $\square$.

Lemma: Given $p_1, \dots, p_n \in D$ & $g_j(z)$ holomorphic on an open $U_j \ni p_j$, then $a_1, \dots, a_n \in \mathbb{Z}_{\geq 0}$

exists $h(z) \in \mathcal{O}(D)$ s.t. $g_j(z) - h(z)$ vanishes at p_j to order at least a_j.

Proof: For each $1 \leq j \leq n$, we consider the Taylor series expansion about p_j of the meromorphic function $G_j := g_j(z) \prod_{i \neq j} (z - p_i)^{-a_i}$

- G_j is holo at p_j, so $G_j = \sum_{r=0}^{\infty} c_j^{(r)} (z - p_j)^r$
- Let φ be a meromorphic function on D with poles exactly at $p_1, \dots, p_n$ st $1 \leq j \leq n$, the Laurent series expansion of φ near p_j is of the form

$$(**) \quad \varphi = \underbrace{\sum_{r=0}^{a_j - 1} \frac{c_j^{(r)}}{(z - p_j)^{a_j - r}}}_{\text{principal part of } \varphi \text{ at } p_j \,=\, \mathrm{Ppal}(\varphi, p_j)} + \text{holomorphic near } p_j$$

(Example: $\varphi = \sum_{j=1}^{n} \mathrm{ppal}(\varphi, p_j)$ works!)

Remark: (**) is the "easy" version of Mittag-Leffler's Theorem (find a meromorphic function φ with poles $p_1, \dots, p_n$, holo away from $p_1, \dots, p_n$ & with prescribed principal parts at each p_j) Hard: replace the finite set with a discrete one.

- Let $\boxed{h(z) = \varphi(z) \prod_{i=1}^{n} (z-p_i)^{a_i}}$. Clearly, $\varphi \in \mathcal{O}(\mathbb{D})$.

Claim: $\forall\, 1 \le j \le n$ $\quad g_j(z) - h(z) = (z-p_j)^{a_j} \cdot (\text{holo at } p_j)$

Pf/ Note: $(z-p_j)^{a_j} \operatorname{Ppal}(\varphi, p_j) = \sum_{r=0}^{a_j-1} c_j^{(r)} (z-p_j)^r$

$$\Rightarrow g_j(z) - h(z) = g_j(z) - (z-p_j)^{a_j} \varphi(z) \prod_{i \ne j} (z-p_i)^{a_i}$$

$$= \prod_{i\ne j} (z-p_i)^{a_i} \Big(g_j(z) \prod_{i \ne j} (z-p_i)^{-a_i} - (z-p_j)^{a_j} \varphi(z) \Big)$$

$$= \prod_{i \ne j} (z-p_i)^{a_i} \Big(\cancel{\sum_{r=0}^{a_j-1} c_j^{(r)} (z-p_j)^r} - \cancel{\sum_{r=0} c_j^{(r)} (z-p_j)^r} + (z-p_j)^{a_j} (\text{holo at } p_j) \Big)$$

$$= (z-p_j)^{a_j} \Big(\prod_{i \ne j} (z-p_i)^{a_i} \cdot \text{holo at } p_j \Big) \qquad \square$$

Remark: The statement of the Theorem and the core of its proof can be phrased using ses on sheaves & long exact sequences in cohomology.

§21.2 The Riemann-Roch Theorem

Theorem R-R Fix X a compact RS of genus g & $D \in \operatorname{Div}(X)$.

Then $H^0(X, \mathcal{O}_D)$ & $H^1(X, \mathcal{O}_D)$ are finite dimensional $\mathbb{C}$-vector spaces &

$$\dim H^0(X,\mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) - \deg D = 1-g$$

Example $D = 0$. Then $\mathcal{O}_D = \mathcal{O}$

$$\underbrace{\dim \underbrace{H^0(X,\mathcal{O})}_{=\mathcal{O}(X)=\mathbb{C}}}_{\substack{=1 \\ \text{(finite)} \checkmark}} - \underbrace{\dim H^1(X,\mathcal{O})}_{\substack{=g \\ \text{(finite)} \checkmark}} - \underbrace{\deg 0}_{=0} = 1-g \qquad \checkmark$$

Proof of R-R We'll do it by establishing finiteness & proving:

$\dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) - \deg D$ is independent of D

($\Rightarrow$ constant $= 1-g$ for $D=0$.)

Pf/ By induction on $|\mathrm{Supp}\, D|$.

- Base case $\mathrm{Supp}\, D = \emptyset$ ie $D=0$ ✓
  - Inductive step: if true for D, show it for $D+[p]$ $\forall p \in X$.

(support may decrease if $\mathrm{ord}_p D = -1$)

For the inductive step we'll need to relate $\mathcal{O}_D$ & $\mathcal{O}_{D\pm[p]}$ using ses on sheaves. This will also prove finiteness. We'll discuss it next time.

Applications Find lower bounds for the dimension of $\mathcal{O}_D(X)$

($\mathcal{O}_D(X) \ni f$ with prescribed $\geq$ order of zeros & $\leq$ order of poles.)

Corollary 1 Fix X a compact RS of genus g & $a \in X$. Then $\exists f \in \mathcal{M}(X) \setminus \mathbb{C}$ where f has at most a pole at a of order $\leq g+1$.

(We've seen $\exists f \in \mathcal{M}(X) \setminus \mathbb{C}$ with a pole at a, but order of this unique pole wasn't bounded. This was Corollary 1 §19.3 (consequence of finiteness thm))

Proof: Apply RR to the divisor $D = (g+1)[a]$

$$\mathcal{O}_D(U) = \{ f \in \mathcal{M}(U) : \ \mathrm{ord}_x(f) \geq -D(x) \ \forall x \in U \}$$

if $a \notin U \Rightarrow \mathcal{O}_D(U) = \mathcal{O}(U)$

if $a \in U \Rightarrow \mathcal{O}_D(U) = \{ f \in \mathcal{M}(U),$ holo away from a & with $\mathrm{ord}_a(f) \geq -(g+1) \}$

(order of a as a pole $\leq g+1$.

- $H^0(X, \mathcal{O}_D) = \mathcal{O}_D(X)$ Want to show: $\dim \geq 2$ ($\mathbb{C} \subseteq \mathcal{O}_D(X)$)

By RR: $\dim H^0(X, \mathcal{O}_D) \geq \dim H^1(X, \mathcal{O}_D) + 1 - g + \deg D$

$$\geq 0 + 1 - g + (1+g) = 2$$

✓ □

Corollary 2: Fix a compact RS X of genus g. Then, $\exists\ f: X \longrightarrow \mathbb{P}^1$ holo branched proper map with at most $g+1$ sheets (degree of $f \leq g+1$)

Proof Use f from Corollary 1: $f \in \mathcal{M}(X) \setminus \mathbb{C}$ gives $f: X \longrightarrow \mathbb{P}^1$ holo & non-constant, so it is proper & degree $= |f^{-1}(\infty)| = \text{order}_a(f) \leq g+1$.

Corollary 3: X compact RS of genus 0. Then $X \simeq \mathbb{P}^1$.

Proof Pick $f: X \longrightarrow \mathbb{P}^1$ from Corollary 2 (proper, non constant, holo map of degree ≤ 1). But $\deg \geq 1$, so degree $= 1$. $f: X \longrightarrow \mathbb{P}^1$ local homeo & proper is biholomorphic.

§21.3 Skyscraper sheaves:

Next Want to relate $\mathcal{O}_D$ & $\mathcal{O}_{D+[p]}$ for $p \in X$ a point ($\rightsquigarrow \mathcal{O}_D$ vs $\mathcal{O}_{D+D'}$) on compact RS

For this, we need the notion of the Skyscraper sheaf $\mathbb{C}_p$.

Definition: Fix X any RS & $p \in X$. The skyscraper sheaf $\mathbb{C}_p$ on X is defined as
$$\mathbb{C}_p(U) = \begin{cases} \mathbb{C} & \text{if } p \in U \\ 0 & \text{else} \end{cases}$$

with restrictions
$$\rho_{UV} : \mathbb{C}_p(U) \longrightarrow \mathbb{C}_p(V) \qquad \Rightarrow \qquad \rho_{UV} = \begin{cases} 0 & \text{if } p \notin V \\ \text{id} & \text{if } p \in V \end{cases}$$
$$(f) \longmapsto \begin{cases} (0) & \text{if } p \notin V \\ (f) & \text{if } p \in V \end{cases}$$

Stalk: $(\mathbb{C}_p)_p = \mathbb{C}$ & $(\mathbb{C}_p)_x = 0 \quad \forall x \neq p$ (hence, the name!)

Proposition: (1) $H^0(X, \mathbb{C}_p) = \mathbb{C}_p(X) = \mathbb{C}$

(2) $H^1(X, \mathbb{C}_p) = 0$

Proof (2) Need to show $H^1(\underline{\mathcal{U}}, \mathbb{C}_p) = 0 \quad \forall\ \underline{\mathcal{U}} = (U_i)_{i \in I}$ open covering of X

($H^1(\underline{\mathcal{U}}, \mathcal{C}_p) \hookrightarrow H^1(X, \mathcal{C}_p)$ by Proposition 2 §16.3)

Claim: $\exists$ a refinement $\underline{\mathcal{V}} < \underline{\mathcal{U}}$ with $\mathcal{V} = (V_k)_{k \in K}$ st p is contained in exactly one V_k.

Pf/ Pick (U, φ) local coord chart at p $\quad U \overset{\varphi}{\simeq} D$, $p \longmapsto 0$ $\quad$ contained in some U_{i_0}

$W = \varphi^{-1}(D_{\frac{1}{2}}(0))$ open nbhd of p & $\overline{W} \subseteq U$ compact

Take $\underline{\mathcal{V}} = \{\underset{V_0}{\underset{\parallel}{U}}\} \cup \{\underset{V_i}{\underbrace{U_i \setminus \overline{W}}} : i \in I\}$ opens $\quad$ & $\tau : K \longrightarrow I$, $\quad 0 \longmapsto i_0$, $\quad I \ni i \longmapsto i$

$\underline{\mathcal{V}}$ covers X (if $x \notin U \Longrightarrow x \in \bigcup_{i \in I}(U_i \setminus U) \subseteq \bigcup_{i \in I}(U_i \setminus \overline{W}) = \bigcup_{i \in I} V_i$.)

$\underline{\mathcal{V}} < \underline{\mathcal{U}}$ via τ $\quad \square$

Consequence: Since $\mathcal{C}_p(V_{k\ell}) = 0$ we get $Z^1(\underline{\mathcal{V}}, \mathcal{C}_p) = \{0\} \Longrightarrow H^1(\underline{\mathcal{V}}, \mathcal{C}_p) = 0$.

($f_{k\ell} \in \mathcal{C}_p(V_{k\ell}) = \{0\}$ $\forall k, \ell$ so $C^1(\mathcal{V}, \mathcal{C}_p) = \{\underline{0}\}$)

But $H^1(\underline{\mathcal{U}}, \mathcal{C}_p) \xhookrightarrow{\tau^{\mathcal{U}}_{\underline{\mathcal{V}}}} H^1(\mathcal{V}, \mathcal{C}_p)$ by Theorem §16.2, so $H^1(\underline{\mathcal{U}}, \mathcal{C}_p) = 0$.

Taking direct limit over $\underline{\mathcal{U}}$, we get $H^1(X, \mathcal{C}_p) = 0$.

§21.4 Exact sequences:

Recall: Given X top space & sheaves $\mathcal{F}$, $\mathcal{G}$ (of ab groups / vector spaces) on X

$\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ morphism of sheaves $\equiv$ $\alpha_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ homomorphism of ab grps / vector spaces compatible with restriction maps for $\mathcal{F}$ & $\mathcal{G}$.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\alpha_U"] \arrow[d, "\rho^{\mathcal{F}}_{U,V}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{G}(U) \arrow[d, "\rho^{\mathcal{G}}_{U,V}"] \\ \mathcal{F}(V) \arrow[r, "\alpha_V"] & \mathcal{G}(V) \end{tikzcd} \qquad \forall\ V \subseteq U \text{ opens.}$$

$\rightsquigarrow$ α induces $\quad \alpha_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ morphism on stalks.

$\qquad [f] \longmapsto [\alpha_V(f)]$ $\quad$ if $f \in \mathcal{F}(V)$ for $x \in V$.

Definition: We say $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ is an isomorphism if $\exists\ \beta : \mathcal{G} \longrightarrow \mathcal{F}$ morphism of sheaves with $\beta \circ \alpha = \mathrm{id}_{\mathcal{F}}$ & $\alpha \circ \beta = \mathrm{id}_{\mathcal{G}}$.

Lemma 1: α is an isomorphism iff α_x is bijective $\forall x$

Proof: $(\Rightarrow)$ Is clear: $(\alpha^{-1})_x = (\alpha_x)^{-1} \quad \forall x.$

$(\Leftarrow)$ We'll show that $\alpha_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is an iso for all open $U \subset X$.

If so $(\alpha^{-1})_U = (\alpha_U)^{-1}$ & the compatibility of the map with the restrictions will follow from that of α.

$$\begin{tikzcd}
\mathcal{G}(U) \arrow[r, shift left, "(\alpha_U)^{-1}"] \arrow[d, "\rho^{\mathcal{G}}_{U,V}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{F}(U) \arrow[l, shift left, "\alpha_U"] \arrow[d, "\rho^{\mathcal{F}}_{U,V}"] \\
\mathcal{G}(V) \arrow[r, shift left, "(\alpha_V)^{-1}"] & \mathcal{F}(V) \arrow[l, shift left, "\alpha_V"]
\end{tikzcd}$$

• α_U is injective: Pick $s \in \mathcal{F}(U)$ with $\alpha_U(s) = 0$ Then $\alpha_x(s_x) = [\alpha_U(s)] = [0] \ \forall x$

Thus $s_x = 0 \quad \forall x.$ This forces $s=0$ ($\forall x \in U \ \exists \ V_x \subseteq U$ open with $x \in V_x$ st $s|_{V_x} = 0$ These sections $s|_{V_x} = 0 \in \mathcal{F}(V_x)$ agree on the overlaps, so by the sheaf axiom on $\mathcal{F}$, since s & $0 \in \mathcal{F}(U)$ satisfy $s|_{V_x} = 0$, we get $s=0$ by the uniqueness)

• α_U is surjective: Pick $t \in \mathcal{G}(U)$ Then, $t_x \in \mathcal{G}_x = \alpha_x(\mathcal{F}_x) \ \forall x \in X$.

Therefore, we can find $U_x \ni x$ open $U_x \subseteq U$ & $f_x \in \mathcal{F}(U_x)$ with $\alpha_x([f_x]) = t_x$

We consider $\alpha_{V_x}(f_x) \in \mathcal{G}(U)$. By construction $[\alpha_{V_x}(f_x)] = [t]$ on $\mathcal{G}_x$.

In particular, shrinking x if needed, we see $\alpha_{V_x}(f_x) = t|_{x}$ (by definition of stalk $\mathcal{G}_x$)

Pick $x, y \in U$ & set $V = U_x \cap U_y$. Then

$$\alpha_V(f_x|_V) = \rho_{U_xV}(\alpha_{U_x}(f_x)) = \rho_{U_xV}(t|_{U_x}) = t|_V = \rho_{U_yV}(t|_{U_y}) = \rho_{U_yV}(\alpha_{U_y}(f_y)) = \alpha_V(f_y|_V)$$

Since α_V is injective, we get $f_x|_V = f_y|_V$.

Conclusion $f_x \in \mathcal{F}(U_x)$ agree on the overlaps so $\exists! \ f \in \mathcal{F}(U)$ with

$f|_{U_x} = f_x \quad \forall x \in U.$

Now: $\alpha_U(f)|_{U_x} = t|_{U_x} \in \mathcal{G}(U_x) \quad \forall x$ & agree on the overlaps

By the sheaf axiom on $\mathcal{G}$: $\alpha_U(f) = t$ $\square$

Definition: α is a monomorphism iff α_x is injective $\forall x$ $(K(\alpha)_x = \{0\}\ \forall x)$

α is an epimorphism iff α_x is surjective $\forall x$ $(B(\alpha)_x = \mathcal{G}_x\ \forall x)$

Lemma 2: If α is a monomorphism, then $\forall U$ open $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective

Pf/ This is part of the proof of Lemma 1.

⚠ The analogous statement for epimorphisms fails!

Example: $X = \mathbb{C}^*$ $\quad \mathcal{F} = \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* = \mathcal{G}$

- At stalks, the map is injective because we have branches of log near each pt (after making a cut)
- We don't have a global log, so $\exp_X : \mathcal{O}(X) \longrightarrow \mathcal{O}^*(X)$ is not surjective ($z \notin \operatorname{im}(\exp_X) \subseteq \mathcal{O}^*(\mathbb{C}^*)$)

Def: A sequence of sheaves $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$ is exact iff for each $x \in X$: $\mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta} \mathcal{H}_x$ is exact (sequence of ab sps/vector spaces)

Lemma 3: If $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ is exact then $\forall U \subseteq X$ open, the sequence $0 \to \mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U) \xrightarrow{\beta_U} \mathcal{H}(U)$ in $\mathcal{C} = Ab$ or $Vect$ exact.

⚠ We lose exactness at $\mathcal{H}$. This is what cohomology will measure

Proof. Exactness at $\mathcal{F}(U)$ holds by Lemma 2

- $\underline{\operatorname{Im} \alpha_U} \subset \underline{\operatorname{Ker} \beta_U}$: Pick $g \in \mathcal{F}(U)$ & set $f = \alpha_U(g) \in \mathcal{G}(U)$.

Want to show $\beta(f) = 0$.

Since $\mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x$ is exact $\forall x$, given $x \in X$ $\exists$ V_x open with

$x \in V \subseteq U$ and $\beta_{V_x}(g|_{V_x}) = \beta(g)|_{V_x} = 0$ $\quad (\beta_x = g_x = \beta_x(\alpha_x(f)) = 0)$

By sheaf axiom $\beta(g)|_{V_x}$ glue to $\beta(g)$ & to 0, so $\beta(g) = 0$.

- $\underline{\operatorname{Ker}\beta_U \subset \operatorname{Im}\alpha_U}$: Fix $g \in \operatorname{Ker}(\beta_U)$ ie $\beta_U(g) = 0$. with $g \in \mathcal{G}(U)$.

Since $\operatorname{Ker}\beta_x = \operatorname{Im}\alpha_x$ $\forall x$ we can find an open covering $(V_x)_{x\in U}$ of U & elements $f_x \in \mathcal{F}(V_x)$ with $\alpha(f_x) = g|_{V_x}$ on V_x

We want to glue $f_x \in \mathcal{F}(V_x)$ to $f \in \mathcal{F}(U)$.

On $V_x \cap V_y$: $\alpha(f_x) = g|_{V_x\cap V_y} = \alpha(f_y)$ ie $\alpha(f_x - f_y) = 0$

Since $\operatorname{Ker}\alpha_V = 0$ $\forall V$ open by Lemma 2 then $f_x - f_y = 0$ on $V_x \cap V_y$

Thus, the sections $f_x \in \mathcal{F}(V_x)$ agree on the overlaps & they glue to $f \in \mathcal{F}(U)$ with $f|_{V_x} = f_x$.

By construction $\alpha(f)|_{V_x} = \alpha(f|_{V_x}) = g|_{V_x}$ $\forall V_x$.

The sheaf axiom on $\mathcal{G}$ gives $\alpha(f) = g$. (gluing is unique!) $\square$

<u>Theorem</u>: X top space. Then, a ses $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ of sheaves on X induce a long exact sequence on cohomology.

$$
\begin{aligned}
0 \to H^0(X,\mathcal{F}) &\xrightarrow{\alpha^0} H^0(X,\mathcal{G}) \xrightarrow{\beta^0} H^0(X,\mathcal{H}) \xrightarrow{\delta} \\
H^1(X,\mathcal{F}) &\xrightarrow{\alpha^1} H^1(X,\mathcal{G}) \xrightarrow{\beta^1} H^1(X,\mathcal{H}) \xrightarrow{\delta} \\
H^2(X,\mathcal{F}) &\xrightarrow{\alpha^2} \cdots
\end{aligned}
$$

(long exact seq for ab sps / v.sp)

⚠ This will not be true if we work with Čech cohomology relative to coverings (unless the coverings are Leray of arbitrary order for $\mathcal{F}$, $\mathcal{G}$ & $\mathcal{H}$!)

We'll see more on this next time!

Example: Fix X compact RS, $D \in \mathrm{Div}(X)$ & $p \in X$

We build a sequence of sheaves of $\mathbb{C}$-vector spaces

$$(*) \quad \boxed{0 \longrightarrow \mathcal{O}_D \xrightarrow{\text{inc}} \mathcal{O}_{D+[p]} \xrightarrow{\beta} \mathbb{C}_p \longrightarrow 0}$$

- $\mathcal{O}_D \subset \mathcal{O}_{D+[p]}$ since $\mathrm{ord}_x([p]) = \begin{cases} 1 & x = p \\ 0 & \text{else} \end{cases}$ so $f \in \mathcal{O}_D(U)$ gives

$$\mathrm{ord}_x(f) \geq -D_{(x)} \geq -(D+[p])_x \quad \forall x \in U.$$

- Definition of β: $\beta_U = 0$ if $p \notin U$

if $p \in U$ build a local chart (V, z) around p in U with $V \xrightarrow{\sim} \mathbb{D}$, $p \longmapsto 0$

$\Rightarrow$ given $f \in \mathcal{O}_{D+p}(U)$, write the Laurent series expansion of f on V:

$$f = \sum_{n=-k-1}^{\infty} c_n z^n \quad \text{with } k = D(p)$$

$$\Rightarrow \beta_U(f) = c_{-k-1} \in \mathbb{C} = \mathbb{C}_p(U).$$

(Chart independent because stalks correspond to Laurent series expansion & transitions become $\mathbb{D} \xrightarrow{\varphi} \mathbb{D}$ biholomorphisms)

- $(\beta \circ \mathrm{inc})_U \equiv 0$ ($c_{-k-1} = 0$ if $f \in \mathcal{O}_D \subset \mathcal{O}_{D+[p]}$)

Lemma 4 The sequence $(*)$ is exact

Proof: We work with stalks.

- If $x \neq p$: $(\mathcal{O}_D)_x = (\mathcal{O}_{D+[p]})_x$ & $(\mathbb{C}_p)_x = 0$ $\quad$ inc_x iso & $\beta_x \equiv 0$
- If $x = p$: $\mathrm{Im}(\mathcal{O}_D \longrightarrow \mathcal{O}_{D+[p]})_p$ $\quad$ $\mathrm{ord}_p(f) \geq -D_{(p)} > -D_{(p)} - 1$

inc_p is injective & β_p is surjective $\forall p$

if $\beta_p(g) = 0 \Rightarrow g = \sum_{n=-k} c_n z^n$ $\quad$ $(c_{-n-1} = 0)$ so $g \in (\mathcal{O}_D)_p$ $\square$