Lecture XXII: The Riemann-Roch Theorem II

Recall Last time we discussed Riemann-Roch & tools we'll need to prove it: skyscraper sheaves; ses of sheaves

- **Theorem R-R** Fix X a compact RS of genus g & $D \in \mathrm{Div}(X)$.
  Then $H^0(X, \mathcal{O}_D)$ & $H^1(X, \mathcal{O}_D)$ are finite dimensional $\mathbb{C}$-vector spaces &
  $$\dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) - \deg D = 1 - g$$

Applications: give lower bounds on dimension of $H^0(X, \mathcal{O}_D) = \mathcal{O}_D(X)$

$\mathcal{O}_D(X)$ = meromorphic functions on X with prescribed zeros of order $\geq -D(x)$
poles —— $< D(x)$

Zeros $= \{x \in \mathrm{Supp}\, D$ with $D(x) < 0\}$
Poles $= \{x \in \mathrm{Supp}\, D$ —— $D(x) > 0\}$

- **Definition**: Fix X any RS & $p \in X$. The skyscraper sheaf $\mathbb{C}_p$ on X i

$U \subseteq X$ open $\quad \mathbb{C}_p(U) = \begin{cases} \mathbb{C} & \text{if } p \in U \\ 0 & \text{else} \end{cases}$; Restrictions $U \subseteq V$ $\quad \rho_{UV} = \begin{cases} 0 & \text{if } p \notin V \\ \mathrm{id}_{\mathbb{C}} & \text{if } p \in V \end{cases}$

Stalk: $(\mathbb{C}_p)_p = \mathbb{C}$ & $(\mathbb{C}_p)_x = 0 \quad \forall x \neq p$

Proposition: (1) $H^0(X, \mathbb{C}_p) = \mathbb{C}_p(X) = \mathbb{C}$
(2) $H^1(X, \mathbb{C}_p) = 0$

- **Definition**: $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ exact $\Leftrightarrow$ $0 \to \mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x \to 0$ is exact $\forall x \in X$

Lemma: If $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$ is exact then $0 \to \mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U) \xrightarrow{\beta_U} \mathcal{H}(U)$ is exact $\forall U \subseteq X$ open.

- **Key short exact sequence**: $\quad 0 \to \mathcal{O}_D \xrightarrow{inc} \mathcal{O}_{D+[p]} \xrightarrow{\beta} \mathbb{C}_p \to 0 \quad (*)$

On U open: (1) If $p \notin U$, $\beta_U \equiv 0$.
(2) If $p \in U$ & $f \in \mathcal{O}_{D+[p]}(U) \subseteq \mathcal{M}(U)$, write its Laurent series expn around p,
say $f = \sum_{j=-k-1}^{\infty} c_j z^j$ with $k = D(p)$ $\Rightarrow$ $\beta_U(f) = c_{-k-1}$

THM: Short exact sequences on sheaves induce long exact sequences on cohomology

TODAY: Proof of Riemann-Roch & discuss the construction of long exact sequence.

§ 22.1 More on ses (*)

Recall: $\mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ morphism of sheaves is epi/mono $\iff$ all $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$ are
if $f \in \mathcal{F}(U)$ $\quad [f]_x \mapsto [\alpha_U(f)]_x$

$\iff \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \to 0 \;/\; 0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ is exact.

Q: How to define $H^1(X,\mathcal{F}) \xrightarrow{\alpha} H^1(X,\mathcal{G})$ from $\alpha : \mathcal{F} \to \mathcal{G}$?

A: Define $H^1(\underline{\mathcal{U}},\mathcal{F}) \xrightarrow{\alpha} H^1(\underline{\mathcal{U}},\mathcal{G})$ for each covering $\underline{\mathcal{U}}$.

Note: $C^k(\underline{\mathcal{U}},\mathcal{F}) \xrightarrow{\alpha} C^k(\underline{\mathcal{U}},\mathcal{G})$ $k=0,1$ & $\alpha \circ \partial = \partial \circ \alpha$ so this

$(f_{\underline{i}}) \longmapsto (\alpha(f_{\underline{i}}))$ for $i_0,\dots i_k$

map descends to $Z^1(\underline{\mathcal{U}},\mathcal{F}) \xrightarrow{\alpha} Z^1(\underline{\mathcal{U}},\mathcal{G})$ & $B^1(\underline{\mathcal{U}},\mathcal{F}) \xrightarrow{\alpha} B^1(\underline{\mathcal{U}},\mathcal{G})$

Conclude: $$\frac{Z^1(\underline{\mathcal{U}},\mathcal{F})}{B^1(\underline{\mathcal{U}},\mathcal{F})} := H^1(\underline{\mathcal{U}},\mathcal{F}) \xrightarrow{\alpha'} H^1(\underline{\mathcal{U}},\mathcal{G}) := \frac{Z^1(\underline{\mathcal{U}},\mathcal{G})}{B^1(\underline{\mathcal{U}},\mathcal{G})}$$

Since the map α is compatible with refinement maps $\tau^{\mathcal{U}}_{\mathcal{V}}$ for $\underline{\mathcal{V}} \leq \underline{\mathcal{U}}$

we can take direct limit over $\underline{\mathcal{U}}$ & get $H^1(X,\mathcal{F}) \xrightarrow{\alpha'} H^1(X,\mathcal{G})$

In particular, these constructions are compatible with compositions so

$$\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \rightsquigarrow (\alpha \circ \beta)' = \beta' \circ \alpha' : H^1(X,\mathcal{F}) \to H^1(X,\mathcal{H})$$

$\rightsquigarrow$ By Theorem, we get a (long) exact sequence from (*)

$$0 \to H^0(X,\mathcal{O}_D) \xrightarrow{inc^0} H^0(X,\mathcal{O}_{D+[p]}) \xrightarrow{\beta^0} H^0(X,\mathbb{C}_p) = \mathbb{C} \text{ by Prop 3 §21.3}$$

$$\xrightarrow{\delta^*} H^1(X,\mathcal{O}_D) \xrightarrow{inc^1} H^1(X,\mathcal{O}_{D+[p]}) \xrightarrow{\beta^1} H^1(X,\mathbb{C}_p) = 0$$

by Prop 3 §21.3

Corollary If $D \leq D'$ on $\mathrm{Div}(X)$ for X compact RS, then

$H^1(X,\mathcal{O}_D) \xrightarrow{inc} H^1(X,\mathcal{O}_{D'})$ is surjective.

__Proof__ Since $D' \geq D$ we can find $p_1, \dots, p_n \in X$ (possibly repeated) with $D' = D + [p_1] + \cdots\cdots + [p_n]$. We induct on n.

- If $n=1$, the statement follows from the exactness of (*) (Lemma 4 §21.4)

For $n>1$ $\quad D'' = D + [p_1] + \cdots + [p_{n-1}]$ $\quad D'' < D'$ &

$$\begin{tikzcd}
H^1(X,\mathcal{O}_D) \arrow[r, "\mathrm{inc}^1"', "\substack{\downarrow \\ \text{surj by (IH)}}"'] \arrow[rr, bend right, "\mathrm{inc}^1"'] & H^1(X,\mathcal{O}_{D''}) \arrow[r, "\mathrm{inc}^1", "\substack{\downarrow \\ \text{surj by} \\ (n=1)\text{ case}}"'] \arrow[d, phantom, "\circlearrowleft"] & H^1(X,\mathcal{O}_{D'}) \\
& {} &
\end{tikzcd}$$

[$C^k(\underline{U},\mathcal{O}_D) \xrightarrow{\mathrm{inc}} C^k(\underline{U},\mathcal{O}_{D''}) \xrightarrow{\mathrm{inc}} C^k(\underline{U},\mathcal{O}_{D'})$ for $k=0,1$ descends to
$Z^1(\underline{U},\mathcal{O}_D) \xrightarrow{\mathrm{inc}} Z^1(\underline{U},\mathcal{O}_{D''}) \xrightarrow{\mathrm{inc}} Z^1(\underline{U},\mathcal{O}_{D'})$ &
$B^1(\underline{U},\mathcal{O}_D) \xrightarrow{\mathrm{inc}} B^1(\underline{U},\mathcal{O}_{D''}) \xrightarrow{\mathrm{inc}} B^1(\underline{U},\mathcal{O}_{D'})$ because $\mathrm{inc} \circ \partial = \partial \circ \mathrm{inc}$
as we discussed above]

Compositions of surjective morphisms are surjective, so we are done. $\square$

§22.2 Proof of Riemann-Roch

__GOAL__: Show that (1) $H^0(X,\mathcal{O}_D)$ & $H^1(X,\mathcal{O}_D)$ are finite dim'l $\forall D \in \mathrm{Div}(X)$

(2) $\dim H^0(X,\mathcal{O}_D) - \dim H^1(X,\mathcal{O}_D) - \deg D$

is independent of D
(How? Show these properties invariant under $D \longrightarrow D \pm [p]$)
This will prove the statement by induction on $|\mathrm{Supp}\, D|$, writing $D = D' - D''$ for $D', D'' \geq 0$. Enough to see what happens under $D \longrightarrow D + [p]$.

- __Base case__ is $\mathcal{O}_0 = \mathcal{O}$. (1) $\mathcal{O}(X) = \mathbb{C}$ & $H^1(X,\mathcal{O})$ are finite dim'l for X compact RS
(2) $\underbrace{\dim \mathcal{O}(X)}_{1} - \underbrace{\dim H^1(X,\mathcal{O})}_{g} - \underbrace{\deg D}_{0} = 1-g$ ✓

For the inductive step, we consider the ses (*) & the associated long exact seq.

Break the long exact seq. into 2 ses using 2 vector spaces:

$$V := \operatorname{Im}\left(H^0(X, \mathcal{O}_{D+[p]}) \to \mathbb{C}\right) \quad \& \quad W := \mathbb{C}/V$$

$\rightsquigarrow$ (1) $0 \to H^0(X,\mathcal{O}_D) \longrightarrow H^0(X,\mathcal{O}_{D+[p]}) \longrightarrow V \longrightarrow 0$

(2) $0 \longrightarrow W \xrightarrow{\delta^*} H^1(X,\mathcal{O}_D) \longrightarrow H^1(X,\mathcal{O}_{D+[p]}) \longrightarrow 0$

Key: $\dim V + \dim W = 1 = \deg(D+[p]) - \deg(D)$

- Exactness of (1) says $\dim V = \dim H^0(X,\mathcal{O}_{D+[p]}) - \dim H^0(X,\mathcal{O}_D)$
  (ie one H^0 is finite if, and only if, the other one is)
- Exactness of (2) says $\dim W = \dim H^1(X,\mathcal{O}_D) - \dim H^1(X,\mathcal{O}_{D+[p]})$
  (ie one H^1 is finite if, and only if, the other one is)
- Conclusion 1: Finiteness of $H^0(X,\mathcal{O}_D)$ & $H^1(X,\mathcal{O}_D)$ is preserved under $D \to D \pm [p]$
- Combining this with key id, we get

$$\dim H^0(X,\mathcal{O}_{D+[p]}) - \dim H^1(X,\mathcal{O}_{D+[p]}) - \deg(D+[p])$$
$$= \dim H^0(X,\mathcal{O}_D) - \dim H^1(X,\mathcal{O}_D) - \deg(D)$$

- Conclusion 2 $\dim H^0(X,\mathcal{O}_D) - \dim H^1(X,\mathcal{O}_D) - \deg D$ is invariant under $D \longmapsto D \pm [p]$

Therefore, this quantity is independent of D by induction. Indeed write $D = D' - D''$ $D', D'' \geq 0$, so $|\operatorname{supp} D'| \leq |\operatorname{Supp} D|$ & $|\operatorname{Supp} D''| < |\operatorname{Supp} D|$
Induct on $\min\{\deg D', \deg D''\}$.

- Base case: either D' or $D'' = 0$. If so, this follows by Conclusion 1 + induction.
- Inductive Step: Say min at D', use IH on $D' - [p]$ for some $p \in \operatorname{Supp} D$ to get
  $\dim H^0(X, \mathcal{O}_{D'-[p]-D''}) < \infty$. But Conclusion 1, $\dim H^0(X, \mathcal{O}_{D'-D''}) < \infty$.

  If min is at D'' use IH on $(D''-[p])$ for some $p \in \operatorname{Supp} D$ to get
  $\dim H^0(X, \mathcal{O}_{D'-D''+[p]}) < \infty$. By Conclusion 1, $\dim H^0(X, \mathcal{O}_{D'-D''}) < \infty$.

- $\dim H^0(X,\mathcal{O}_D) - \dim H^1(X,\mathcal{O}_D) - \deg D = 1-g-0$ for $D=0$.

Then Conclusion 2 + writing $D = D'-D''$ with $D' \geq D''$ allows us to go

from $D'-[p]-D''$ to $D'-D''$ if $\deg D'>0$ is $\min\{\deg D', \deg D''\}$ $(p\in \operatorname{Supp} D')$
$D'-(D''-[p])$ to $D'-D''$ — $\deg D''>0$ ———— $(p\in \operatorname{Supp} D'')$

Induction on $\min\{\deg D', \deg D''\}$ proves the statement (the base case is $D=D'\geq 0$ or $D=-D''\leq 0$ which we can do by induction on $\deg D$ using Conclusion 2.

$\square$

§ 22.3 Long exact sequences on Cohomology:

Theorem: X top space. Then, a ses $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ of sheaves on X induce a long exact sequence on cohomology.

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{F}) \arrow[r, "\alpha^0"] & H^0(X,\mathcal{G}) \arrow[r, "\beta^0"] & H^0(X,\mathcal{H}) \arrow[dll, "\delta"'] \\
 & H^1(X,\mathcal{F}) \arrow[r, "\alpha^1"] & H^1(X,\mathcal{G}) \arrow[r, "\beta^1"] & H^1(X,\mathcal{H}) \arrow[dll, "\delta"'] \\
 & H^2(X,\mathcal{F}) \arrow[r, "\alpha^2"] & \cdots &
\end{tikzcd}$$

(long exact seq of ab grps/v.sp)

⚠ This will not be true if we work with Čech cohomology relative to coverings (unless the coverings are Leray of arbitrary order for $\mathcal{F}$, $\mathcal{G}$ & $\mathcal{H}$!)
The issue is not just with the connecting maps. It's also exactness at $H^k(X,\mathcal{G})$.

① How to define α^k, β^k? Work with open coverings & take direct limit.
Fix $\underline{\mathcal{U}} = (U_i)_{i\in I}$ open covering on X. Then $\alpha: \mathcal{F} \to \mathcal{G}$ induces

$$C^k(\underline{\mathcal{U}},\mathcal{F}) \xrightarrow{\alpha^k} C^k(\underline{\mathcal{U}},\mathcal{G})$$
$$(f_{\underline{i}}) \longmapsto (\alpha(f_{\underline{i}}))_{\underline{i}}$$

By construction $\alpha^k \circ \partial = \partial \circ \alpha^{k-1}$ so α^k descends to cohomology.
Same goes for $C^k(\underline{\mathcal{U}},\mathcal{G}) \xrightarrow{\beta^k} \check{C}^k(\underline{\mathcal{U}},\mathcal{H})$

• Exactness at $H^0(X,\mathcal{F})$ & $H^0(X,\mathcal{G})$ was seen last time (Take $U = X$ in Lemma 3 §21.4 : $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$ is exact. $\forall U \subseteq$ open)

• Exactness at $H^1(X,\mathcal{G})$:

$\operatorname{Im}\alpha^1 \subset \operatorname{Ker}\beta^1$ Consider a covering $\underline{\mathcal{U}}$ & set $U = U_i \cap U_j$ in Lemma 3 §21.4

$$\mathcal{F}(U_{ij}) \xrightarrow{\alpha} \mathcal{G}(U_{ij}) \xrightarrow{\beta} \mathcal{H}(U_{ij}) \quad \text{is exact}$$

So $\beta \circ \alpha(Z^1(\underline{\mathcal{U}},\mathcal{F})) = 0$. This descends to Čech cohomology relative to the covering $\underline{\mathcal{U}}$. by def of α^1 & β^1.

$$\Rightarrow 0 = \varinjlim_{\underline{\mathcal{U}}} \beta^{\wedge}\left(\alpha^{\wedge}\left(\frac{Z^1(\underline{\mathcal{U}},\mathcal{F})}{B^1(\underline{\mathcal{U}},\mathcal{F})}\right)\right) = \beta^1\left(\varinjlim_{\underline{\mathcal{U}}} \alpha^1 (H^1(\underline{\mathcal{U}},\mathcal{F}))\right)$$
$$= \beta^1 \circ \alpha^1\left(\varinjlim_{\underline{\mathcal{U}}} H(\underline{\mathcal{U}},\mathcal{F})\right) = \beta^1 \circ \alpha^1 (H(X,\mathcal{F}))$$

(same is true $\forall k$: $\operatorname{Im}\alpha^k \subset \operatorname{Ker}\beta^k$)

$\operatorname{Ker}\beta^1 \subset \operatorname{Im}\alpha^1$ Fix $\eta \in \operatorname{Ker}\beta^1$ & let $(g_{ij}) \in Z^1(\underline{\mathcal{U}},\mathcal{G})$ a representative for some $\underline{\mathcal{U}} = (U_i)_{i\in I}$. Then $\exists$ cochain $(h_i) \in C^0(\underline{\mathcal{U}},\mathcal{H})$ s.t

$$\beta(g_{ij}) = \partial h = h_j - h_i. \in \mathcal{H}(U_{ij})$$

Need to replace g_{ij} by something in $\operatorname{Ker}\beta$. If we had $\beta_{U_j}^{-1}(h_j), \beta_{U_i}^{-1}(h_i) \neq \emptyset$ that would solve it. Unfortunately, β_{U_j}, β_{U_i} need not be surjective, but the surjectivity we do have is at the level of stalks. This forces us to work with a refinement $\underline{\mathcal{V}} \leq \underline{\mathcal{U}}$, $\underline{\mathcal{V}} = \{V_x\}_{x\in X}$ to be determined:

Given $x \in X$ pick $i = \tau(x)$ with $x \in U_i$ ($\rightsquigarrow$ map $X \xrightarrow{\tau} I$)

We know $\mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x$ is surjective, so $\exists\, x \in V_x \subseteq U_{\tau x}$ open & $g_x \in \mathcal{G}(V_x)$ with $\beta_{V_x}(g_x) = h_{\tau x}|_{V_x} \in \mathcal{H}(V_x)$

We use $\{V_x\}_x$ to build a refinement $\underline{\mathcal{V}} < \underline{\mathcal{U}}$ $\quad (\underline{\mathcal{V}} := \{V_x\}_{x\in X})$
(The refinement function is τ)

Set $\boxed{\tilde{g}_{xy} = g_{\tau x,\tau y}|_{V_x\cap V_y}} \in \mathcal{G}(V_x\cap V_y) \rightsquigarrow \tilde{g} \in C^1(\underline{\mathcal{V}},\mathcal{G})$

<u>Claim</u>: $\tilde{g} \in Z^1(\underline{\mathcal{V}},\mathcal{G})$ & $[\tilde{g}] = \eta$ in $H^1(\underline{\mathcal{V}},\mathcal{G})$ via $\tau^{\mathcal{U}}_{\mathcal{V}}$.

Pf/ $(\partial\tilde{g})_{xyz} = g_{\tau y\tau z} - g_{\tau x\tau z} + g_{\tau y \tau z} = (\partial g)_{\tau x\tau y\tau z}|_{V_x\cap V_y\cap V_z} \overset{g\in Z^1(\underline{\mathcal{U}},\mathcal{G})}{=0}$ on $V_x\cap V_y\cap V_z \subseteq U_{xyz}$

<u>Advantage</u>: $\beta(\tilde{g}_{xy}) = h_{\tau y}|_{V_x\cap V_y} - h_{\tau x}|_{V_x\cap V_y}$ on $V_x\cap V_y \subseteq U_{\tau x,\tau y}$, $\tilde{g}_{xy} = \tau^{\mathcal{U}}_{\mathcal{V}}(g_{ij})$.

(so $[\tilde{g}] = [g] = \eta$ in $H^1(\underline{\mathcal{V}},\mathcal{G})$) & <u>now</u> $h_{\tau y}|_{V_x\cap V_y}$ & $h_{\tau x}|_{V_x\cap V_y}$ lie in $\operatorname{Im}\beta_{V_x\cap V_y}$.

Set $\boxed{\Psi_{xy} = \tilde{g}_{xy} + g_x|_{V_x\cap V_y} - g_y|_{V_x\cap V_y} = \tilde{g}_{xy} - (\partial g)_{\tau x\tau y}|_{V_x\cap V_y}} \in \mathcal{G}(V_x\cap V_y)$

<u>Claim</u>: $\beta(\Psi_{xy}) = 0$ on $\mathcal{H}(V_x\cap V_y)$

$$\beta(\Psi_{xy}) = \beta(\tilde{g}_{xy}) - \beta(\partial g) = \beta(g_{\tau x,\tau y}) - \beta_{V_y}(g_{\tau y}|_{V_y}) + \beta_{V_x}(g_{\tau x}|_{V_x})$$
$$= h_{\tau y}|_{V_{yx}} - h_{\tau x}|_{V_{xy}} - h_{\tau y}|_{V_{yx}} + h_{\tau x}|_{V_{xy}} = 0$$

<u>Claim</u>: $\Psi \in Z^1(\underline{\mathcal{V}},\mathcal{G})$ & $[\Psi] = [\tilde{g}]$ in $H^1(\underline{\mathcal{V}},\mathcal{G})$

Pf/ . $\tilde{g}$ & $\partial g|_{V_x\cap V_y} \in Z^1(\underline{\mathcal{V}},\mathcal{G})$ abelian gp wrt $+$, so $\Psi \in Z^1(\underline{\mathcal{V}},\mathcal{G})$

- $\Psi - \tilde{g} = \partial(-g)|_{V_x\cap V_y} \in B^1(\underline{\mathcal{V}},\mathcal{G})$.

$\Rightarrow$ By Lemma 3.22.4 (for $U = V_x\cap V_y$), $\exists\ f_{xy} \in \mathcal{F}(V_x\cap V_y)$ with

$\alpha(f_{xy}) = \Psi_{xy}$ (because $\beta_{V_x\cap V_y}(\Psi_{xy}) = 0$)

<u>Claim</u>: $f_{xy} \in Z^1(\underline{\mathcal{V}},\mathcal{F})$

Pf/. α is injective so $0 \to \mathcal{F}(V_x \cap V_y \cap V_z) \xrightarrow{\alpha} \mathcal{G}(V_x \cap V_y \cap V_z)$ is exact

$\alpha(\partial f)_{xyz} = \partial(\alpha(f_{xy}))_{xyz} = (\partial \psi)_{xyz} = 0$ forces $\partial f = 0$, ie $(f_{xy}) \in Z^1(\underline{\mathcal{V}}, \mathcal{F})$ □

Conclude: $\xi = [f] \in H^1(\underline{\mathcal{V}}, \mathcal{G})$ satisfies $\alpha(\xi) = [\psi] = [\tilde{g}] = \eta$ in $H^1(\underline{\mathcal{V}}, \mathcal{G})$

Observation: The same proof strategy works for higher cohomology

⚠ It shows why we need to work with the direct limit & not with the Čech cohomology groups relative to a fixed covering

② Connecting homomorphism? $\delta^*: H^0(X, \mathcal{H}) \longrightarrow H^1(X, \mathcal{F})$

$h \longmapsto ?$

Fix $h \in H^0(X, \mathcal{H}) = \mathcal{H}(X)$. Since $\beta_x : \mathcal{G}_x \to \mathcal{H}_x$ is surjective

$\exists\, g_x \in \mathcal{G}_x$ with $\beta_x(g_x) = h_x$

$\Rightarrow \exists\, U_x$ open with $x \in U_x$ & $g_{U_x} \in \mathcal{G}(U_x)$ with $\beta(g_{U_x}) = h|_{U_x}$.

⇝ Collect these opens into a covering $\underline{\mathcal{U}} = (U_i)_{i \in I}$ $(I = X)$. & write g_i for g_{U_i}

$\Rightarrow g_i$ satisfy $\beta(g_i) = h|_{U_i \cap U_j} = \beta(g_j)$ on $U_i \cap U_j$

$\Rightarrow \beta(g_i - g_j) = 0$ on $U_i \cap U_j$

By Lemma 3 §21.4, $\exists\, f_{ij} \in \mathcal{F}(U_i \cap U_j)$ with $\alpha(f_{ij}) = g_j - g_i$ on $U_i \cap U_j$.

Claim: $f \in Z^1(\underline{\mathcal{U}}, \mathcal{F})$

Pf/ $\alpha(\partial f)_{i_0 i_1 i_2} = \partial(\alpha(f))_{i_0 i_1 i_2} = \alpha(f_{i_1 i_2}) - \alpha(f_{i_0 i_2}) + \alpha(f_{i_0 i_1})$

$= (g_{i_2} - g_{i_1}) - (g_{i_2} - g_{i_1}) + (g_{i_1} - g_{i_0}) = 0$

By Lemma 2 §21.4, $(\partial f)_{i_0 i_1 i_2} = 0$ $(\forall i_0 i_1 i_2)$ ie $f \in Z^1(\underline{\mathcal{U}}, \tilde{\mathcal{F}})$.

Claim 2: $\delta^* h = [f] \in H^1(X, \tilde{\mathcal{F}})$ is independent of choices.

Pf/ (1) Take common refinements & use $H^1(\underline{\mathcal{U}}, \tilde{\mathcal{F}}) \xrightarrow{\tau^{\mathcal{U}}_{\mathcal{V}}} H^1(\mathcal{V}, \mathcal{F})$ if $\underline{\mathcal{V}} < \underline{\mathcal{U}}$ to work with a fixed refinement (given 2 $\underline{\mathcal{U}}, \underline{\mathcal{U}}'$ set $\underline{\mathcal{V}}$ = common refinement of $\underline{\mathcal{U}}$ & $\underline{\mathcal{U}}'$)

(2) Any other choice of g_i will be of the form $g_i + \alpha(p_i)$ with $p_i \in \mathcal{F}(U_i)$

$\Rightarrow \exists\ \tilde{f}_{ij} \in \tilde{\mathcal{F}}(U_i \cap U_j)$ with $\alpha(\tilde{f}_{ij}) = g_j + \alpha(p_j) - (g_i + \alpha(p_i))$

Thus $\alpha(\tilde{f}_{ij} + p_i - p_j) = g_j - g_i = \alpha(f_{ij})$ on U_{ij} $\forall i,j$

Since $\alpha_{U_{ij}}$ is injective, we get $\tilde{f}_{ij} + p_i - p_j = f_{ij}$

Now $\tilde{f} \in Z^1(\underline{\mathcal{U}}, \tilde{\mathcal{F}})$ by our earlier claim.

Need to show $[\tilde{f}] = [f]$ in $H^1(\underline{\mathcal{U}}, \tilde{\mathcal{F}}) \longrightarrow H^1(X, \tilde{\mathcal{F}})$.

But this is easy since $\tilde{f} = f + \partial p$ $(p = (p_i) \in C^0(\underline{\mathcal{U}}, \tilde{\mathcal{F}}))$

Claim 3: $\operatorname{Im} \beta^0 \subset \operatorname{Ker}(\delta^*)$ & $\operatorname{Im}(\delta^*) \subset \operatorname{Ker}(\alpha')$ by construction

(Use $g_i = g|_{U_i}$ for $\beta_x(g) = h$) (Use $[\alpha(f)] = [\partial(g_i)] = 0$ if $[f] = \delta^*(h)$)

Claim 4: $\operatorname{Ker}(\delta^*) \subset \operatorname{Im} \beta^0$

Pf/ If $\delta^*(h) = 0$ we can find $f \in Z^1(\underline{\mathcal{U}}, \tilde{\mathcal{F}})$ with $\delta^* h = [f] = [0]$

By def of δ^*: $\exists\, g_i = g_{U_i} \in \mathcal{G}(U_i)$ with $\beta(g_{U_i}) = h$ on U_i.

$\alpha(f_{ij}) = g_j - g_i$ on U_{ij}

But $[f] = 0 \Rightarrow \exists (F_i) \in C^0(\underline{\mathcal{U}}, \tilde{\mathcal{F}})$ with $f_{ij} = \partial F = F_j - F_i$ $\forall i,j$

Take $\tilde{g}_i = g_i - \alpha(F_i)$ $\forall i$

Then $\tilde{g}_i = \tilde{g}_j$ on U_{ij} because $g_j - g_i = \alpha(f_{ij}) = \alpha(F_j - F_i)$

We glue $\tilde{g}_i$ to a section $\tilde{g} \in \mathcal{G}(X)$ with $\tilde{g}|_{U_i} = \tilde{g}_i$

$\Rightarrow \beta(\tilde{g})|_{U_i} = \beta(g_i - \alpha(F_i)) = \beta(g_i) = h$ ie $h \in \operatorname{Im} \beta^0(X)$.

<u>Claim 5:</u> $\operatorname{Ker}\alpha' \subset \operatorname{Im}(\delta^*)$

Pf/ Fix $\xi \in \operatorname{Ker}\alpha'$. Pick covering $\underline{\mathcal{U}}$ with $\xi \in H^1(\underline{\mathcal{U}}, \mathcal{F})$ ($\& \alpha(\xi) = 0$ in $H^1(\underline{\mathcal{U}}, \mathcal{G})$)

Pick $f \in Z^1(\underline{\mathcal{U}}, \mathcal{F})$ with $[f] = \xi$

Since $\alpha'(\xi) = 0$, $\exists\, g = (g_i) \in C^0(\underline{\mathcal{U}}, \mathcal{G})$ with $\alpha(f_{ij}) = \partial g = (g_j - g_i)$ in U_{ij}

$\Rightarrow \quad 0 = \beta\alpha(f_{ij}) = \beta(g_j) - \beta(g_i)$ in U_{ij}

By Sheaf axiom $\exists\, h \in \mathcal{H}(X) = H^0(X, \mathcal{H})$ with $h_{|U_i} = \beta(g_i)\ \forall i$

The construction of δ^* shows $\delta^*(h) = [f]$

<u>Note</u>: Same proofs will work for higher cohomology groups (we don't use $\tau^{\mathcal{U}}_{\mathcal{V}}$ is injective on H^1, & we have induced refinement maps $H^k(\underline{\mathcal{U}}, -) \xrightarrow{\tau^{\mathcal{U}}_{\mathcal{V}}} H^k(\underline{\mathcal{V}}, -)$)