Lecture XXIII: Serre Duality I

Recall . X compact R.S, then genus $= \dim H^1(X,\mathcal{O})$
- $K = (\omega)$ for $\omega \in \mathcal{M}^{(1)}(X)$ is a canonical divisor on X.

Riemann-Roch: For X compact RS & $D \in \mathrm{Div}(X)$, we have $H^0(X,\mathcal{O}_D), H^1(X,\mathcal{O}_D)$ are finite dimensional & $\dim H^0(X,\mathcal{O}_D) - \dim H^1(X,\mathcal{O}_D) - \deg D = 1-g$.

Remark: $\exists\, f \in \mathcal{M}(X) \setminus \mathbb{C}$ with a pole (consequence of R-R) $\Rightarrow df \in \mathcal{M}^{(1)}(X)$ gives a non-constant element in $\mathcal{M}^{(1)}(X)$ & so a non-trivial $K = (df)$

Lemma 1: For $\omega_1, \omega_2 \in \mathcal{M}^{(1)}(X)$ we have $(\omega_1) \sim (\omega_2)$ are linearly equivalent canonical divisors

Lemma 2: $D = D' + (f)$ then $\mathcal{O}_{D'} \xrightarrow[\cdot f]{\sim} \mathcal{O}_D$ isomorphism of sheaves.

Next goal: Give alternative ways to compute the genus of compact R.S.

① [Serre duality] $g = \dim H^0(X, \Omega)$ $\Omega =$ sheaf of holomorphic 1-forms

② $\deg K = 2g-2$.

dual vector space

For ① we'll build an isomorphism $H^1(X,\mathcal{O}_D) \simeq H^0(X,\Omega_D)^{\vee}$ (above: $D=0$ case)

to be defined ($\Omega_0 = \Omega$)

The map will be defined using Residues of smooth 2-forms, or equivalently, of meromorphic 1-forms.

§23.1 The sheaves Ω_D & Serre duality

Recall: $\Omega = \mathrm{Ker}\left(\mathcal{E}_X^{(1,0)} \xrightarrow{d} \mathcal{E}_X^{(2)}\right)$ sheaf of holomorphic 1-forms on X = R.S.

On (U,z) chart: $f\,dz \overset{d}{\longmapsto} \frac{\partial f}{\partial \bar z}\, dz \wedge d\bar z$, so $f\,dz \in \Omega(U) \Leftrightarrow f \in \mathcal{O}_{(z)}$.

Assume X is a R.S. & $D \in \mathrm{Div}(X)$

Definition: We define a sheaf Ω_D on X via

$$\Omega_D(U) = \{\omega \in \mathcal{M}^{(1)}(U) : (\omega) \geq -D \text{ on } U\}$$ (analog of $\mathcal{O}_D$)

On charts (U,z) : $\omega \in \mathcal{M}^{(1)}(U) \iff \omega = f\,dz$ $\quad f \in \mathcal{M}(\mathbb{D})$ $\quad \mathrm{ord}_x(\omega) = \mathrm{ord}_{z(x)} f$

$U \xrightarrow[z]{\sim} \mathbb{D}$

Example: $\Omega_0(U) = \{\omega \in \mathcal{M}^{(1)}(U) : (\omega) \geq 0 \text{ on } U\}$

On a chart (V,z) with $V \subseteq U$ $\quad \underset{|_V}{\omega} = f\,dz$ & $\mathrm{ord}_{z(x)}(f) \geq 0 \;\forall x$, so f is holomorphic on V, ie $\omega_{|V} \in \Omega(V)$. Converse is clear.

Conclude: $\Omega_0 = \Omega$

Proposition: If $D = D' + (f)$ for $f \in \mathcal{M}(X) \setminus \{0\}$, then $\Omega_{D'} \xrightarrow[\cdot f]{\sim} \Omega_D$ is an isomorphism of sheaves.

Lemma: (1) $H^0(\mathbb{P}^1, \Omega) = \Omega(\mathbb{P}^1) = \{0\}$

(2) $H^1(\{U_0, U_\infty\}, \Omega) = \mathbb{C}\langle \frac{dz}{z} \rangle$

(3) $H^1(\mathbb{P}^1, \Omega) = \mathbb{C}$

Proof (1) $\Omega(\mathbb{P}^1) = \{\omega \in \mathcal{M}^{(1)}(\mathbb{P}^1) \;\; (\omega) \geq 0\}$ $\quad$ Assume $\exists\, \eta \in \Omega(\mathbb{P}^1) \setminus \{0\}$

We use the open covering $\{U_0, U_\infty\}$ of $\mathbb{P}^1$ to describe η

$\eta|_{U_0} = f\,dz$ $\quad f$ holo on $U_0 = \mathbb{C}$ $\qquad \eta|_{U_\infty} = g\,dw$ $\quad g$ holo on $U_\infty \simeq \mathbb{C}$

Gluing $\quad z = \frac{1}{w}$ $\qquad g(w)\,dw = f\left(\frac{1}{w}\right)\frac{-dw}{w^2} = \frac{f(\frac{1}{w})}{w^2}\,dw$ on $U_0 \cap U_\infty \simeq \mathbb{C}^*$

g holo near $\infty \Rightarrow f\left(\frac{1}{w}\right)\frac{1}{w^2}$ holo near 0.

$\Rightarrow f$ has a pole at ∞ forcing f to be a polynomial $f \neq 0$

$\mathrm{ord}_0 f\left(\frac{1}{w}\right) - 2 = \mathrm{ord}_\infty(g) \geq 0 \Rightarrow \mathrm{ord}_0 f\left(\frac{1}{w}\right) - 2 = -\deg f - 2 \geq 0$ Contra!

(2) Pick $\underline{\mathcal{U}} = \{U_0, U_\infty\}$ cover of $\mathbb{P}^1$ $\Rightarrow Z^1(\underline{\mathcal{U}}, \Omega) = C^1(\underline{\mathcal{U}}, \Omega)$ (no triple intersection)

$\Omega(U_0 \cap U_\infty) = \{f(z)\,dz \quad f \in \mathcal{O}(\mathbb{C}^*)\}$

$B^1(\underline{\mathcal{U}}, \Omega) = \partial C^0(\underline{\mathcal{U}}, \Omega)$ $\quad$ & $C^0(\underline{\mathcal{U}}, \Omega) = \{(\underset{\in \Omega(U_0)}{g_0(z)\,dz}, \underset{\in \Omega(U_\infty)}{g_\infty(w)\,dw})\}$

$\Rightarrow \partial g = g_\infty(w)\,dw - g_0(z)\,dz \underset{\text{on } U_\infty \cap U_0 = \mathbb{C}^*}{=} (-z^{-2}\underbrace{g_\infty(1/z)}_{\text{holo on } U_0} - \underbrace{g_0(z)}_{\text{holo on } U_0})\,dz$ on $U_0 \cap U_\infty$

Now, $f \in \mathcal{O}(\mathbb{C}^*)$. Write $f(z) = \sum_{n=2}^{\infty} f_{-n} z^{-n} + f_{-1} z^{-1} + \sum_{\ell=0}^{\infty} f_\ell z^\ell$ $\quad$ Laurent expn about 0 ROC $= \infty$

$$\Rightarrow f(z)\,dz = f_1 z^{-1} dz + \partial(g) \quad \text{where} \quad \begin{aligned} g_\infty(w) &= -\sum_{n=2}^{\infty} f_{-n} w^{n-2} \in \mathcal{O}(U_\infty) \\ g_0(z) &= -\sum_{\ell=0}^{\infty} f_\ell z^\ell \in \mathcal{O}(U_0) \end{aligned}$$

[<u>Check</u>: $g_\infty(\frac{1}{z}) = -(\sum_{n=2}^{\infty} f_{-n} z^{-n}) z^2 \Rightarrow g_\infty(\frac{1}{z})(-\frac{dz}{z^2}) = \sum_{n=2}^{\infty} f_{-n} z^{-n}$]

$$\Rightarrow [f(z)\,dz] = [f_1 \tfrac{dz}{z}] = f_1 [\tfrac{dz}{z}] \quad \text{in } H^1(\underline{\mathcal{U}}, \Omega)$$

(3) <u>Exercise</u> (Hint: Show $H^1(D_R(0), \Omega) = 0 \;\forall\, 0 < R \leq \infty$. by Dolbeault's Thm)

<u>Theorem</u> (Serre duality) Fix X compact RS & $D \in \mathrm{Div}(X)$. Then $\exists$ a non-deg. pairing

"Residue map"

$$\begin{array}{ccccc} H^0(X, \Omega_{-D}) \times H^1(X, \mathcal{O}_D) & \longrightarrow & H^1(X, \Omega) & \xrightarrow{\text{"Res"}} & \mathbb{C} \\ (\omega) \times [g] & \longmapsto & [g\omega] & \longmapsto & \mathrm{Res}([g\omega]) \end{array}$$

<u>Corollary 1</u>: $\dim_{\mathbb{C}} H^0(X, \Omega_{-D}) = \dim_{\mathbb{C}} H^1(X, \mathcal{O}_D) < \infty$ (by RR) $\forall D$

<u>Proof</u>: $H^0(X, \Omega_{-D}) \simeq H^1(X, \mathcal{O}_D)^\vee$ & $\dim H^1(X, \mathcal{O}_D) < \infty$ by RR

so $H^1(X, \mathcal{O}_D) \simeq H^1(X, \mathcal{O}_D)^\vee$ (identify dual bases).

$\Rightarrow H^0(X, \Omega_{-D}) \simeq H^1(X, \mathcal{O}_D)$ as finite dimensional $\mathbb{C}$-vector spaces, so their dimensions agree. □

<u>Corollary 2</u>: $\dim H^0(X, \Omega) = \text{genus}(X)$ for any compact RS X

<u>Proof</u>: Take $D = 0$ so $\Omega_{-0} = \Omega$. Then, Corollary 1 says

$$\dim H^0(X, \Omega) = \dim H^1(X, \mathcal{O}) = \text{genus}(X)$$ □

<u>Corollary 3</u>: (Reformulation of Riemann-Roch) Fix X compact RS of genus g, $D \in \mathrm{Div}(X)$

Then, $\dim H^0(X, \mathcal{O}_{-D}) - \dim H^0(X, \Omega_D) = 1 - g - \deg D$

(use $-D$ in RR instead of D + Corollary 1)

<u>Next</u>: understand the pairing in Serre's duality Thm

§23.2 Ω_D vs $\mathcal{O}_D$ & Serre's pairing: Fix X = compact R.S.

Proposition: Given $\omega \in \mathcal{M}^{(1)}(X) \setminus \{0\}$ & $D \in \mathrm{Div}(X)$ we have an isomorphism of sheaves $\varphi: \mathcal{O}_{D+K} \xrightarrow{\sim} \Omega_D$ $\quad f \longmapsto f\omega$ where $K = (\omega)$

Proof: Given $x \in X$, pick a local chart (U,z) around x. $\quad U \xrightarrow[z]{\sim} \mathbb{D}$, $x \longmapsto 0$

Write $\omega|_U = g\,dz$ with $g \in \mathcal{M}(\mathbb{D})$

φ is Well-defined: $f \in \mathcal{O}_{D+K}(V)$, $x \in V$. Pick a local chart (U,z) around x with $U \subseteq V$. Then: $\mathrm{ord}_x(f) \geq -(D+K)_{(x)} = -D_{(x)} - \mathrm{ord}_x(\omega)$

Write $k = D(x)$ & $p = \mathrm{ord}_x(\omega) = \mathrm{ord}_x(g)$ $\quad$ Pick chart $U \subseteq V$

$\Rightarrow f\omega|_U = (fg)dz$ & $\mathrm{ord}(fg) = \mathrm{ord}(f) + \mathrm{ord}(g) \geq -k-p+p = -k$

$\Rightarrow f\omega|_U \in \Omega_D(U)$ $\quad$ These sections agree on overlaps (they are determined by the same formula $f\omega$!) so they give a section $f\omega \in \mathcal{O}_D(V)$.

- Enough to check φ is iso on stalks. WLOG, given $x \in X$ & the chart (U,z), we can assume assume $\mathrm{Supp}\, D \cap U \subseteq \{x\}$ (Supp D is finite because X is compact.)

Injective: Fix $f \in \mathcal{O}_{D+K,x}$ with $\varphi(f) = 0 \in \Omega_{D,x}$

$[f\omega]_x = [fg\,dz]_0$ says Laurent power series exp of fg at 0 is 0.

Since $g \not\equiv 0$ ($\mathbb{D}$ connected), we conclude $[f]_x = 0$.

Surjective. Fix $\eta \in \Omega_{D,x}$ so $\exists$ chart (U,z) around x & $h \in \mathcal{O}_D(U)$ with $\eta = [h\,dz]$. In particular $h \in \mathcal{M}(U)$, $h|_{U^*} \in \mathcal{O}(U^*)$ & $\mathrm{ord}_x(h) \geq -D(x)$
(We pick U so that $\mathrm{Supp}\, D \cap U \subseteq \{x\}$.)

Then $(\omega|_U) \sim (h\,dz)$ on $\mathrm{Div}(U)$, so $\exists f \in \mathcal{M}(U)$ with $\eta = (f) + (\omega|_U)$

$$\mathrm{ord}_y(f) = \mathrm{ord}_y(\eta) - \mathrm{ord}_y(\omega|_U) = \begin{cases} 0 - \mathrm{ord}_y(K) & \text{if } y \neq x \\ \geq -D(x) - \mathrm{ord}_x(K) & \text{if } y = x \end{cases}$$

So $f \in \mathcal{O}_{D+K}(U)$ & $f\omega|_U = h\,dz$ on U

Conclude: $[f\omega]_x = \eta$ ie $\varphi(f) = \eta$ $\square$

Obs: We can reinterpret the previous result as a map of sheaves

$$\Omega_{-K} \oplus \mathcal{O}_{D+K} \longrightarrow \Omega_D \qquad D = -K + D + K$$
$$(\omega) = K \geq -(-K)$$

On U open:
$$\Omega_{-K}(U) \times \mathcal{O}_{D+K}(U) \longrightarrow \Omega_D(U)$$
$$\omega \quad f \longmapsto f\omega$$

Remark: Same ideas of Proposition produce a map of sheaves via multiplication

$$\Omega_{-D'} \oplus \mathcal{O}_D \longrightarrow \Omega_{-D'+D}$$
$$(\omega, f) \longmapsto f\omega$$

In particular, for $D' = -D$ we get $\Omega_{-D} \times \mathcal{O}_D \longrightarrow \Omega$

Serre's pairing: The multiplication map $\Omega_D \times \mathcal{O}_D \longrightarrow \Omega$ induces a bilinear map $\Psi : H^0(X, \Omega_{-D}) \times H^1(X, \mathcal{O}_D) \longrightarrow H^1(X, \Omega)$

How? $H^0(X, \Omega_{-D}) = \Omega_{-D}(X) = \{\omega \in \mathcal{M}^{(1)}(X) : (\omega) \geq D\}$

$$H^1(X, \mathcal{O}_D) = \varinjlim_{\underline{\mathcal{U}}} H^1(\underline{\mathcal{U}}, \mathcal{O}_D) = \varinjlim_{\underline{\mathcal{U}}} Z^1(\underline{\mathcal{U}}, \mathcal{O}_D) / B^1(\underline{\mathcal{U}}, \mathcal{O}_D).$$

Given $\xi \in H^1(X, \mathcal{O}_D)$ pick $\underline{\mathcal{U}}$ & a representative $(f_{ij}) \in Z^1(\underline{\mathcal{U}}, \mathcal{O}_D)$ with $[f_{ij}] = \xi \in H^1(\underline{\mathcal{U}}, \mathcal{O}_D)$.

We define $\Psi_{\underline{\mathcal{U}}} : H^0(X, \Omega_{-D}) \times H^1(\underline{\mathcal{U}}, \mathcal{O}_D) \longrightarrow H^1(\underline{\mathcal{U}}, \Omega)$
$$\omega \times [f_{ij}] \longmapsto [f_{ij}\,\omega|_{U_{ij}}]$$

Claim 1: $(f_{ij}\,\omega|_{U_{ij}}) \in Z^1(\underline{\mathcal{U}}, \Omega)$

Pf/ $f_{ij}\,\omega|_{U_{ij}} \in \Omega(U_{ij})$ by the Remark $(-D + D = 0)$

$$\partial(f_{ij}\,\omega|_{U_{ij}})_{i_0 i_1 i_2} = f_{i_1 i_2}\,\omega|_{U_{i_1 i_2}} - f_{i_0 i_2}\,\omega|_{U_{i_0 i_2}} + f_{i_0 i_1}\,\omega|_{U_{i_0 i_1}} \quad \text{on } U_{i_0 i_1 i_2}$$

$$= (\partial f)_{i_0 i_1 i_2}\,\omega|_{U_{i_0 i_1 i_2}} \underset{f\in Z^1(\underline{U},\mathcal{O}_D)}{=} 0 \quad \text{on } U_{i_0 i_1 i_2}$$

(Key: ω is a global section of Ω_{-D})

Claim 2: $\Psi_{\underline{U}}$ is well-defined & bilinear

Pf/ • The bilinear condition is clear.

• Well-def: $[f_{ij}]=[g_{ij}]$ in $H^1(\underline{U},\mathcal{O}_D) \Leftrightarrow \exists\, h\in C^0(\underline{U},\mathcal{O}_D)$ with $g_{ij} = f_{ij} + (\partial h)_{ij}\ \forall ij$

$$(\partial h)_{ij}\,\omega|_{U_{ij}} = (h_j - h_i)\,\omega|_{U_{ij}} = h_j\,\omega|_{U_{ij}} - h_i\,\omega|_{U_{ij}} = \partial(\underbrace{h_i\,\overbrace{\omega|_{U_i}}^{\omega_i\in\Omega_{-D}(U_i)}}_{\in\Omega(U_i)})$$

$h_i \in \mathcal{O}_D(U_i)$

so $(h_i\,\omega|_{U_i})_i \in C^0(\underline{U},\Omega)$

$$\Rightarrow [g_{ij}\,\omega|_{U_{ij}}] = [f_{ij}\,\omega|_{U_{ij}} + (\partial h)\,\omega|_{U_{ij}}] = [f_{ij}\,\omega|_{U_{ij}} + \partial(h_i\,\omega|_{U_i})]$$

$$= [f_{ij}\,\omega|_{U_{ij}}] \quad \text{in } H^1(\underline{U},\Omega)$$

Claim 3: $\Psi_{\underline{U}}$ is compatible with refinements ie $\Psi_{\underline{V}} = t^{\underline{U}}_{\underline{V}} \circ \Psi_{\underline{U}}$

Consequence: $\Psi_{\underline{U}}$ descends to a bilinear map $\Psi: H^0(X,\Omega_{-D})\times H^1(X,\mathcal{O}_D)\to H^1(X,\Omega)$ $\square$

Next: we need to define a linear map $H^1(X,\Omega) \xrightarrow{\text{Res}} \mathbb{C}$. We'll do this next time.

Serre duality Thm: The Serre pairing + Res yield an isomorphism

$$\boxed{\begin{array}{rcl} i_D: H^0(X,\Omega_{-D}) & \longrightarrow & (H^1(X,\mathcal{O}_D))^\vee \\ \omega & \longmapsto & (\xi \longmapsto \text{Res}(\xi\omega)) \end{array}}$$

$\forall D\in \text{Div}(X)$ (X compact RS)

§23.4 Consequences of Serre duality:

Theorem 1: X compact RS & $D \in \mathrm{Div}(X)$. Then $H^0(X, \mathcal{O}_{-D}) \simeq H^1(X, \Omega_D)^\vee$

Proof: Fix $\omega_0 \in \mathcal{M}^{(1)}(X) \setminus \{0\}$ & set $K = (\omega_0)$. By Proposition §23.2 we have

$$\mathcal{O}_{D+K} \simeq \Omega_D \quad \& \quad \mathcal{O}_{-D} \simeq \Omega_{-D-K} \qquad (D' = -D-K)$$

$$\Rightarrow \quad H^0(X, \mathcal{O}_{-D}) \simeq H^0(X, \Omega_{-D-K}) \quad \& \quad H^1(X, \Omega_D)^\vee \simeq H^1(X, \mathcal{O}_{D+K})^\vee$$

- Serre duality applied to $D' = -D-K$ gives

$$H^0(X, \Omega_{-D-K}) \simeq H^1(X, \mathcal{O}_{D+K})^\vee,$$

so we get the desired isomorphism

Corollary 1: $\dim H^1(X, \Omega) = \dim H^0(X, \mathcal{O}) = 1$

Proof. $H^0(X, \mathcal{O}) = \mathcal{O}(X) = \mathbb{C}$ because X is compact

- Use $D = 0$ in previous Thm to get the equality of dimensions

Theorem 2: The degree of any canonical divisor on a compact RS of genus g is $2g-2$.

Proof Fix $\omega \in \mathcal{M}^{(1)}(X) \setminus \{0\}$ & set $K = (\omega)$. By Riemann-Roch applied to K we get $\dim H^0(X, \mathcal{O}_K) - \dim H^1(X, \mathcal{O}_K) = 1 - g + \deg K$.

By Proposition §22.2 $\Omega \simeq \mathcal{O}_K$ (take $D = 0$) so

$$1 - g + \deg K = \underbrace{\dim H^0(X, \Omega)}_{= \dim H^1(X, \mathcal{O})^\vee = g \text{ by Serre duality}} - \underbrace{\dim H^1(X, \Omega)}_{= 1 \text{ by Coro } 1} = g - 1$$

$\Rightarrow \deg K = 2g - 2$ ✓ $\square$

Corollary 2: For any rank-2 lattice $\Lambda \subseteq \mathbb{C}$, the R.S. $\mathbb{C}/\Lambda$ has genus 1.

Proof dz is a 1-form on $\mathbb{C}$ & it descends to a 1-form ω on $X = \mathbb{C}/\Lambda$

- ω is holomorphic & has no zeros on X, so $\deg \omega = 0$
- But $0 = \deg \omega = 2g-2$ by Theorem 2, so $g=1$.

<u>Obs:</u> This says the topological genus of $\mathbb{C}/\Lambda$ matches the algebraic/geometric genus

($\mathbb{C}/\Lambda$ is homeo to $S^1 \times S^1$)