Lecture XXIV: Serre Duality II

Recall: Given X RS & $D \in \mathrm{Div}(X)$, we define the sheaves $\Omega_D \subseteq \mathcal{M}^{(1)}$ & $\mathcal{O}_D \subseteq \mathcal{M}$ on X

$U \subseteq X$ open : $\Omega_D(U) = \{\omega \in \mathcal{M}^{(1)}(U) : (\omega) \geq -D_{|U}\}$; $\mathcal{O}_D(U) = \{f \in \mathcal{M}(U) : (f) \geq -D_{|U}\}$

Example $\Omega_0 = \Omega$, $\mathcal{O}_0 = \mathcal{O}$.

Lemma 1: $D = D' + (f)$ for $f \in \mathcal{M}(X) \smallsetminus \{0\}$ $\Rightarrow$ $\Omega_{D'} \xrightarrow[\cdot f]{\sim} \Omega_D$ iso of sheaves

$\mathcal{O}_{D'} \xrightarrow[\cdot f]{\sim} \mathcal{O}_D$

Lemma 2: (1) $H^0(\mathbb{P}^1, \Omega) = \{0\}$ vs $H^0(\mathbb{P}^1, \mathcal{O}) = \mathbb{C}$

(2) $H^1(\mathbb{P}^1, \Omega) = \mathbb{C}\langle \frac{dz}{z} \rangle$ vs $H^1(\mathbb{P}^1, \mathcal{O}) = 0$

(3) $H^1(\mathbb{D}_R(0), \Omega) = H^1(\mathbb{D}_R(0), \mathcal{O}) = 0$ $\forall\ 0 < R \leq \infty$ (by Dolbeault's Th)

Prop: If $\omega \in \mathcal{M}^{(1)}(X) \smallsetminus \{0\}$ & $D \in \mathrm{Div}(X)$ we have $\mathcal{O}_{D+K} \xrightarrow[\cdot \omega]{\sim} \Omega_D$

This extends to a map of sheaves $\Omega_{-D'} \oplus \mathcal{O}_D \longrightarrow \Omega_{D-D'}$ & for $D' = D$ we get a bilinear map $\Psi : H^0(X, \Omega_{-D}) \times H^1(X, \mathcal{O}_D) \longrightarrow H^1(X, \Omega) \xrightarrow{\mathrm{Res}} \mathbb{C}$

Formula on a cover $\mathfrak{U}$: $(\ \omega\ ,\ [f_{ij}]\ ) \longmapsto [f_{ij}\omega]$ ↑ [TODAY]

Serre duality Thm: The Serre map + Res yield an isomorphism

$$\boxed{\begin{array}{rcl} i_D : H^0(X, \Omega_{-D}) & \longrightarrow & H^1(X, \mathcal{O}_D)^\vee \\ \omega & \longmapsto & (\xi \longmapsto \mathrm{Res}(\xi\omega)) \end{array}}$$

$\forall\ D \in \mathrm{Div}(X)$ (X compact RS.)

Consequences:
1. $\dim H^1(X, \Omega) = 1$
2. $\deg K = 2g - 2$ K canonical divisor on a genus g compact RS.
3. $H^0(X, \mathcal{O}_{-D}) \simeq H^1(X, \Omega_D)$

§24.1 More consequences of Serre duality

Theorem: Assume X is a compact RS of genus g & fix $D \in \mathrm{Div}(X)$.
Then $H^1(X, \mathcal{O}_D) = 0$ if $\deg D > 2g-2$

Proof Fix $\omega \in \mathcal{M}^{(1)}(X) \smallsetminus \{0\}$ & $K = (\omega)$ By Proposition §23.2, we have

$\mathcal{O}_{-D+K} \simeq \Omega_{-D} \implies H^0(X, \mathcal{O}_{-D+K}) \simeq H^0(X, \Omega_{-D}) \underset{\downarrow}{\simeq} H^1(X, \mathcal{O}_D)^\vee$

Serre duality

If $\deg D > 2g-2 \underset{\substack{\downarrow \\ \text{Thm 2}}}{=} \deg K \implies \deg(-D+K) < 0$

Exercise (HW) $H^0(X, \mathcal{O}_{D'}) = 0$ if $\deg D' < 0$.

Conclude: $H^1(X, \mathcal{O}_D) = 0$ □

Corollary: Assume X is a compact RS. Then, $H^1(X, \mathcal{M}) = 0$.

Proof: We pick an open covering $\underline{\mathcal{U}} = (U_i)_{i \in I}$ of X & show $H^1(\underline{\mathcal{U}}, \mathcal{M}) = 0$

Fix $\xi \in H^1(\underline{\mathcal{U}}, \mathcal{M})$ & a representative $(f_{ij}) \in Z^1(\underline{\mathcal{U}}, \mathcal{M})$ with $[f_{ij}] = \xi$

Claim: $\exists\ \underline{\mathcal{V}} \leq \underline{\mathcal{U}}$ refinement & $(g_{k\ell}) = (f_{\tau k \tau \ell}|_{V_{k\ell}}) \in Z^1(\underline{\mathcal{V}}, \mathcal{M})$

where the total number of poles of all $g_{k\ell}$'s is finite.

Pf/ Refine by local charts (U_x, φ). $U_x \xrightarrow{\sim} \mathbb{D}$ & take $V_x = \varphi_x^{-1}(\mathbb{D}_{1/2})$

with $\begin{array}{l} x \in U_x \subseteq U_{i(x)=i} \\ y \in U_y \subseteq U_{i(y)=j} \end{array}$ & f_{ij} is the extension of g_{xy} from V_{xy} to U_{ij}.

Since the poles of f_{ij} are discrete & $\varphi(\overline{\mathbb{D}_{1/2}})$ is compact, we see that g_{xy} has finitely many poles.

Since X is compact, we can assume $\underline{\mathcal{V}}$ is a finite open cover (just take a finite subcover of $\underline{\mathcal{V}}$). □

$\implies$ gathering the poles of all g_{ij} we can build a divisor $D \geq 0$ of $\deg D > 2g-2$

with $g_{k\ell} \in \mathcal{O}_D(V_{k\ell})$ $\forall k, \ell$ (no conditions on zeroes of $g_{k\ell}$ & $D(x) \geq -\mathrm{ord}_x(g_{k\ell})$ $\forall x$ pole of $g_{k\ell}$)

So $g_{k\ell} \in Z^1(\underline{\mathcal{V}}, \mathcal{O}_D) \underset{\text{Thm}}{=} B^1(\underline{\mathcal{V}}, \mathcal{O}_D) \subseteq B^1(\underline{\mathcal{V}}, \mathcal{M})$

Conclude $[g_{k\ell}] = [\tau^{\mathcal{U}}_{\mathcal{V}}(f_{ij})] = 0$ so $\xi = [f_{ij}] = 0$ because $\tau^{\mathcal{U}}_{\mathcal{V}}$ is injective (by Theorem §16.2) □

§ 24.2 Serre duality for $\mathbb{P}^1$: Fix $D \in \mathrm{Div}(\mathbb{P}^1)$

Define: $H^1(\mathbb{P}^1, \Omega) = \mathbb{C}\langle \frac{dz}{z} \rangle \xrightarrow{\mathrm{Res}} \mathbb{C}$ picks up the coeff.

GOAL: Compute bases of $H^0(\mathbb{P}^1, \Omega_{-D})$, $H^1(\mathbb{P}^1, \mathcal{O}_D)$ that are dual to each other via the Serre pairing

· We know: $\mathcal{O}_D$ & Ω_{-D} only depend on class of D modulo linear equivalence.

Proposition: On $\mathbb{P}^1$ $D \sim D' \iff \deg D = \deg D'$

Proof: $(\Rightarrow)$ $\deg(f) = 0$ & $\deg : \mathrm{Div}\,\mathbb{P}^1 \longrightarrow \mathbb{Z}$ is group hom.

$(\Leftarrow)$ It's enough to show $D \sim \deg D\,[p]$ for any pt $p \in \mathbb{P}^1$ (*)

Moreover, it's enough to show the statement is true for effective divisors.

Write $D = D' - D''$ where $D', D'' \geq 0$ & $\mathrm{Supp}\, D \cap \mathrm{Supp}\, D'' = \emptyset$

$\left(D' = \sum_{\substack{x \\ D(x) \geq 0}} D(x)[x] \right.$ & $\left. D'' = \sum_{\substack{x \\ D(x) < 0}} -D(x)[x] \right)$. Assume (*) true for D', D'',

we write $D' = (f') + \deg D'[p]$ & $D'' = (f'') + \deg D''[p]$ for $f', f'' \in \mathcal{K}(X) \setminus \{0\}$.

$\Rightarrow D = D' - D'' = (f') + \deg D'[p] - (f'') + \deg D''[p] = (f'/f'') + \deg D[p]$,

ie $D \sim \deg D[p]$.

We prove (*) for $D \geq 0$ by induction on $d = |\mathrm{Supp}\, D|$

Base cases: $d = 0$ means $D = 0 = 0[p]$ so they are lin equiv

$d = 1$ $D = a[q]$ $(a \geq 1)$ Claim $[q] \sim [p]$ via $f = \frac{z-q}{z-p}$ in $\mathbb{P}^1$.

$\Rightarrow a[q] \sim a[p]$ via (f^a)

Inductive Step: We write $D = \sum_{i=0}^{d} a_i[q_i]$ $a_i \geq 1$.

$$\Rightarrow D = \sum_{i=0}^{d} a_i[q_i] \underset{d=1 \text{ case}}{\sim} \sum_{i=0}^{d} a_i[p] = \left(\sum_{i=0}^{d} a_i\right)[p] = \deg D\,[p]$$

Conclusion: We need only compute Serre pairing for $D = n[0]$. for $n \in \mathbb{Z}$.

We compute $H^0(\mathbb{P}^1, \Omega_{-D})$ & $H^1(\mathbb{P}^1, \mathcal{O}_D)$ separately.

1 $\boxed{\text{Computation of } H^0(\mathbb{P}^1, \Omega_{-n[0]}):}$

Write any $\eta \in H^0(\mathbb{P}^1, \Omega_{-n[0]})$, ie $(\omega) \geq -(-n[0]) = n[0]$, by its restrictions to U_0 & U_∞.

$\eta|_{U_0} = z^n g_0(z)\,dz$ on U_0 with $g_0 \in \mathcal{O}(U_0)$

$\eta|_{U_\infty} = g_\infty(w)\,dw$ on U_∞ $g_\infty \in \mathcal{O}(U_\infty)$

In particular: $-w^{-n-2} g_0(\frac{1}{w}) = g_\infty(w)$ on $\mathcal{O}(U_0 \cap U_\infty) = \mathcal{O}(\mathbb{C}^*)$

$\Rightarrow g_0$ is holo at ∞ so it must be a polynomial

We have: $\mathrm{ord}_0(\eta) = n + \mathrm{ord}_0 g_0(z)$

$\mathrm{ord}_\infty(\eta) = -n-2 + \mathrm{ord}_\infty g_0 = -n-2 - \deg g_0$

We have 2 cases to analyze: $n \geq -1$ & $n \leq -2$ (write $n = -2-\ell$ for $\ell \geq 0$)

Lemma 1: $H^0(\mathbb{P}^1, \Omega_{-n[0]}) = \{0\}$ if $n \geq -1$.

Proof: $\mathrm{ord}_0(\eta) \geq n \geq -1$. from U_0

$\mathrm{ord}_\infty(\eta) = -n-2 - \deg g_0 \geq 0$

If $\eta \neq 0 \Rightarrow g_0 \neq 0$ so $\deg g_0 > 0$. We get a contradiction

since $-2 \geq -2 - \deg g_0 \geq n \geq -1$ is impossible!

Conclude $H^0(\mathbb{P}^1, \Omega_{-n[0]}) = 0$. $\square$

Lemma 2: $H^0(\mathbb{P}^1, \Omega_{-(2+\ell)[0]})$ has dimension $\ell+1$ $\forall \ell \geq 0$

Proof $\mathrm{ord}_\infty(\eta) = (2+\ell) - 2 - \deg g_0 = \ell - \deg g_0 \geq 0$ from U_∞ forces

η holo on ∞.

$\deg g_0 \leq \ell$

From here we get a basis for $H^0(\mathbb{P}^1, \Omega_{-(2+l)[0]}) = \{ z^{-l-2+i} dz \}_{i=0}^{l}$

① $\boxed{\text{Computation of } H^1(\mathbb{P}^1, \mathcal{O}_{n[0]}):}$

Remark By Thm §21.2, we know that open covers by local charts homeo to unit discs are Leray coverings of 1^{st} order for $\mathcal{O}_D$. The same proof works for charts homeo to discs $D_R(0)$ for $0 < R \leq \infty$ (we only used Supp D is finite & Dolbeault's Thm, which is valid for $D_R(0)$).

This remark ensures $H^1(\mathbb{P}^1, \mathcal{O}_{n[0]}) = H^1(\underline{\mathcal{U}}, \mathcal{O}_{n[0]})$ for $\underline{\mathcal{U}} = \{U_0, U_\infty\}$

$$H^1(\mathbb{P}^1, \mathcal{O}_{n[0]}) = \frac{Z^1(\underline{\mathcal{U}}, \mathcal{O}_{n[0]})}{B^1(\underline{\mathcal{U}}, \mathcal{O}_{n[0]})} = \frac{C^1(\underline{\mathcal{U}}, \mathcal{O}_{n[0]})}{B^1(\underline{\mathcal{U}}, \mathcal{O}_{n[0]})}$$

no triple intersection so $C^2(\underline{\mathcal{U}}, \mathcal{O}_{-n[0]}) = 0$.

- $C^1(\underline{\mathcal{U}}, \mathcal{O}_{n[0]}) = \mathcal{O}(U_0 \cap U_\infty)$ because $\mathrm{Supp}(n[0]) \cap U_0 \cap U_\infty = \emptyset$.

$\Rightarrow$ We need to determine $B^1(\underline{\mathcal{U}}, \mathcal{O}_{n[0]}) = \partial C^0(\underline{\mathcal{U}}, \mathcal{O}_{n[0]})$

- $C^0(\underline{\mathcal{U}}, \mathcal{O}_{n[0]}) = (z^{-n} f_0, f_\infty)$ $\quad f_0 \in \mathcal{O}(U_0)$ & $f_\infty \in \mathcal{O}(U_\infty)$
- Idea: Identify portions of the Laurent series expansion about 0 of any $g \in \mathcal{O}(U_0 \cap U_\infty)$ as $\partial(z^{-n} f_0, f_\infty)$ for appropriate f_0, f_∞.

Again, we break our analysis in 2 cases: $n \geq -1$ & $n \leq -2$ (write $n = -2-l$, $l \geq 0$)

Lemma 3: $H^1(\mathbb{P}^1, \mathcal{O}_{n[0]}) = \{0\}$ if $n \geq -1$

Proof • If $n = -1$ write $g = \underbrace{\sum_{k=0}^{\infty} g_{-k} z^{-k}}_{= f_\infty \in \mathcal{O}(U_\infty)} + \underbrace{z \sum_{k=0}^{\infty} g_{k+1} z^{k}}_{= -f_0 \in \mathcal{O}(U_\infty)}$

- If $n \geq 0$ $\quad g = \underbrace{\sum_{k=-n-1}^{\infty} g_{-k} z^{-k}}_{= f_\infty} + \underbrace{z^{-n} \sum_{k=0}^{\infty} g_k z^{k+n}}_{= -f_0}$

Lemma 4: $H^1(\mathbb{P}^1, \mathcal{O}_{-(2+\ell)[0]})$ has dimension $\ell+1$ $\forall \ell \geq 0$

Proof $C^0(\underline{U}, \mathcal{O}_{-(2+\ell)[0]}) = (z^{2+\ell} f_0, f_\infty)$

Write $g = \underbrace{\sum_{k=0}^{\infty} g_k z^k}_{= f_\infty} + g_1 z + \cdots + g_{1+\ell} z^{1+\ell} + \underbrace{z^{2+\ell} \sum_{k=2+\ell}^{\infty} g_K z^{K-(2+\ell)}}_{= -f_0}$

$\Rightarrow [g] = [g_1 z + \cdots + g_{1+\ell} z^{1+\ell}] = g_1[z] + \cdots + g_{1+\ell}[z^{1+\ell}]$.

so $H^1(\mathbb{P}^1, \mathcal{O}_{-(2+\ell)[0]})$ has dim $1+\ell$ & basis $\{[z^{1+\ell-j}]\}_{j=0}^{\ell}$

Q: What's the Serre map $H^0(\mathbb{P}^1, \Omega_{-n[0]}) \times H^1(\mathbb{P}^1, \mathcal{O}_{n[0]}) \longrightarrow H^1(\mathbb{P}^1, \Omega)$?

$$H^1(\mathbb{P}^1, \Omega) = \mathbb{C}\langle \tfrac{dz}{z} \rangle$$

Case 1: $n \geq -1$ we get $0 \times 0 \longrightarrow \mathbb{C}$ is the 0-map.

Case 2: $n = -2-\ell$ ($\ell \geq 0$) we get.

$$\Psi \quad \mathbb{C}\langle [z^{-\ell-2+i} dz] \rangle_{i=0}^{\ell} \times \mathbb{C}\langle [z^{1+\ell-j}] \rangle_j^{\ell} \longrightarrow \mathbb{C}\langle \left[\tfrac{dz}{z}\right] \rangle$$

on basis elements: $(z^{-\ell-2+i} dz, \; z^{1+\ell-j}) \longmapsto [z^{-\ell-2+i+1+\ell-j}] = [z^{i-j-1}]$

So we get $= \begin{cases} \left[\frac{dz}{z}\right] & \text{if } i=j \\ 0 & \text{else} \end{cases}$

The Residue map $H^1(\mathbb{P}^1, \Omega) \to \mathbb{C}$ picks up the coefficient of $\left[\frac{dz}{z}\right]$.

So we see that $\mathrm{Res} \circ \Psi$ is a perfect pairing. This is Serre duality for $\mathbb{P}^1$

Next tasks: ① Define $\mathrm{Res}: H^1(X, \Omega) \xrightarrow{\cong \mathbb{C} \text{ (after the fact)}} \mathbb{C}$ without using $\dim H^1(X, \Omega) = 1$

② Relate the construction to computation of residues of 1-forms.
(e.g. $\mathrm{Res}(\frac{dz}{z}) = 1$.

§24.3 Residues in $\mathcal{E}^{(2)}(X)$:

Recall the ses of sheaves on any RS X

$$0 \longrightarrow \Omega \longrightarrow \mathcal{E}^{(1,0)} \xrightarrow{d} \mathcal{E}^{(2)} \longrightarrow 0 \tag{$*$}$$

On stalks $a \in X$: use (U,z) local chart around a small enough.

$$\Omega_a = \{ f\, dz \ : \ f \text{ holo on } \mathbb{D} \}$$

$$\mathcal{E}^{(1,0)}_a = \{ f\, dz \ : \ f \text{ smooth on } \mathbb{D} \}$$

$$\mathcal{E}^{(2)}_a = \{ g\, dz \wedge d\bar{z} : g \text{ ——— } \}$$

$$d_a(f\, dz) = \frac{\partial f}{\partial \bar{z}}\, dz \wedge d\bar{z}.$$

- Exact at $\mathcal{E}^{(2)}$ by Dolbeault's Theorem.
- Exact at Ω by CR equations
- Exact at $\mathcal{E}^{(1,0)}$ by $\frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow f \in \mathcal{O}(\mathbb{D})$.

The ses $(*)$ induces a long exact sequence:

$$0 \longrightarrow H^0(X,\Omega) \xrightarrow{\text{inc}} H^0(X,\mathcal{E}^{(1,0)}) \xrightarrow{d^0} H^0(X,\mathcal{E}^{(2)}) \xrightarrow{\delta}$$

$$H^1(X,\Omega) \xrightarrow{\text{inc}} \underset{\underset{0}{\shortparallel}}{H^1(X,\mathcal{E}^{(1,0)})} \xrightarrow{d^1} \underset{\underset{0}{\shortparallel}}{H^1(X,\mathcal{E}^{(2)})}$$

by Corollary §16.4

$$\Rightarrow \boxed{H^1(X,\Omega) \underset{\delta}{\cong} \mathcal{E}^{(2)}(X) \big/ d\left(\mathcal{E}^{(1,0)}(X)\right) = \operatorname{Ker} \delta}.$$

Definition: Assume X is a compact RS. Then:

$$\operatorname{Res} : \mathcal{E}^{(2)}(X) \longrightarrow \mathbb{C} \qquad \text{is a } \mathbb{C}\text{-linear map}$$

$$\omega \longmapsto \frac{1}{2\pi i} \iint_X \omega$$

As a consequence of Stokes's Thm: $\mathrm{Res}(d\eta)=0 \quad \forall \eta \in \mathcal{E}^{(1,0)}(X)$. (Theorem 2 §18.3)

Consequence: Res descends to a linear map

$$\mathrm{Res}: H^1(X,\Omega) \longrightarrow \mathbb{C}$$
$$\xi \longmapsto \mathrm{Res}(\omega) \quad \text{if } [\bar{\delta}\omega]=\xi.$$

This is well-defined since $\mathrm{Res}|_{d\mathcal{E}^{(1,0)}} \equiv 0$.

Q How do we see $\mathrm{Res}\left(\left[\frac{dz}{z}\right]\right)=1$ for $X=\mathbb{P}^1$? **A**: Identify $\omega\in\mathcal{E}^{(2)}(X)$ with a collection of 1-forms & define their residues.

§24.4 Residues on $\mathcal{E}^{(1,0)}$

To define residues, we need to allow singularities on 1-forms in $\mathcal{E}^{(1,0)}$

Fix $Y\subseteq X$ open set in a RS, $a\in Y$ & $\omega\in\mathcal{E}^{(1,0)}(Y\setminus\{a\})$ holomorphic

Fix (U,z) a coord chart of a with $U\subseteq Y$ & $z(a)=0$. Write

$\omega = f\,dz$ for $f_{\circ z^{-1}}\in\mathcal{O}(U\setminus\{a\})\simeq\mathcal{O}(\mathbb{D}\setminus\{0\})$

Write the Laurent series expansion of f: $\quad f=\sum_{n=-\infty}^{\infty} c_n z^n$ about 0 in $\mathbb{D}$

Def: • If $c_n=0 \ \forall n<0$, then a is a removable sing of ω

• If $c_n=0 \ \forall n<-k$ & $a_{-k}\neq 0$, then a is a pole of ω of order k

• If $c_n\neq 0$ for infinitely many $n<0$, a is an essential sing of ω

Def: $\mathrm{Res}_a(\omega) = c_{-1}$ ($\Rightarrow$ it's additive)

Lemma 1: The definition is independent on the choice of chart (U,z).

Proof: Idea: if we pick a different chart, the coefficient of f changes, but not the one from $f\,dz$. We separate $f\,dz$ into an exact form + something with easy residue. We compute the residues of each part separately

Claim 1: If $\omega = dg$ for $g \in \mathcal{O}(V \setminus \{a\})$ then $\operatorname{Res}_a(dg) = 0$ & so it's indep of choices

Pf/ Write $g \circ \varphi^{-1}(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ for the Laurent series exp of g about a

Then $dg = \left(\sum_{n=-\infty}^{\infty} n c_n z^{n-1}\right) dz$ so coeff of z^{-1} is 0.

No matter the chart, the answer is 0.

Claim 2: If ω is a holomorphic 1-form at a, then $\operatorname{Res}_a(\omega) = 0$

Pf/ We write $\omega = f\,dz$ $f \in \mathcal{O}(V)$. Then $f\,dz = \sum_{n=0}^{\infty} c_n z^n dz$ & $\operatorname{Res}_a(\omega) = 0$

It's indep of choices because for any other coord chart $\omega = h\,dz$ with $h \in \mathcal{O}(U)$.

Claim 3: If $\varphi \in \mathcal{O}(V)$ & $\varphi \in \mathfrak{m}_a \setminus \mathfrak{m}_a^2$, then $\operatorname{Res}_a(\varphi^{-1} d\varphi) = 1$ & so it is independent of the choice of chart.

Pf/ Write $\varphi = z h$ for $h \in \mathcal{O}(V)$ $h(a) \neq 0$.

Then $d\varphi = h\,dz + z\,dh$ & $\frac{d\varphi}{\varphi} = \frac{dz}{z} + \frac{dh}{h}$

Since $h(a) \neq 0$ we have $\frac{dh}{h}$ is a holomorphic 1-form at a & so $\operatorname{Res}_a\left(\frac{dh}{h}\right) = 0$ by Claim 2.

$\Rightarrow \operatorname{Res}_a \frac{d\varphi}{\varphi} = \operatorname{Res}_a \frac{dz}{z} + 0 = 1$ (indep of choice by construction) □

- The general statement is obtained by combining Claims 1 & 3.

Write $\omega = f\,dz$ with $f = \sum_{n=-\infty}^{\infty} c_n z^n$

Consider $g = \sum_{n=-\infty}^{-2} \frac{c_n}{n+1} z^{n+1} + \sum_{n=0}^{\infty} \frac{c_n}{n+1} z^{n+1} \in \mathcal{O}(V \setminus \{a\})$

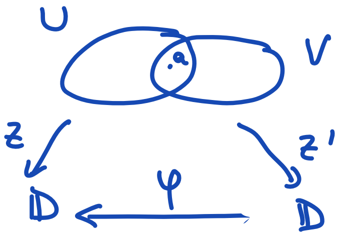
Then $\omega = dg + c_{-1} z^{-1} dz$ $\quad \operatorname{Res}_a \omega = \operatorname{Res}_a dg + c_{-1} \operatorname{Res}_0\left(\frac{dz}{z}\right)$

$= 0 + c_{-1} \cdot 1 = c_{-1}.$

Next, we confirm this decomposition behaves well under coordinate changes.

Pick (V, z') $\quad z' : V \longrightarrow \mathbb{D}$ another coord chart around a.

Then $\omega = f\,dz = h\,dz'$



This means $H_{(z')} = h \circ (z')^{-1} dz' = \underbrace{f \circ z^{-1}}_{F(z)} \circ \varphi_{(z')}\, d(\varphi_{(z')}) = F \circ \varphi_{(z')}\, \varphi' dz'$

The decomposition of $F(z) = \left(g + \frac{c_{-1}}{z}\right) dz$ becomes

$$H_{(z')} dz' = (g \circ \varphi)\, \varphi'\, dz' + c_{-1} \frac{\varphi'}{\varphi} dz' = dg + c_{-1} \frac{\varphi'}{\varphi} dz'$$

But $\mathrm{Res}_a\, dg = 0$ so $\mathrm{Res}_a H_{(z')} = c_{-1}\, \mathrm{Res}_a \frac{\varphi'}{\varphi} dz'$

Since φ is a biholomorphism $\varphi_{(z')} = a z' + \mathcal{O}((z')^2)$ & $a \neq 0$

By claim 3 $\mathrm{Res}_a \frac{\varphi'}{\varphi} dz' = 1$.

Conclusion: $\mathrm{Res}_a\, h\,dz' = c_{-1}$ as we wanted to show. $\square$

Alternative proof: $\mathrm{Res}_a \omega = \frac{1}{2\pi i} \int_{\odot_a} \omega$ for a circle small enough around a.

Residue Theorem: Assume X is compact & consider n distinct points $a_1, \dots, a_n \in X$.
Assume $\omega \in \mathcal{E}^{(1)}(X \setminus \{a_1, \dots, a_n\})$ is holomorphic. Then, $\sum_{j=1}^{n} \mathrm{Res}_{a_j} \omega = 0$.

For a proof, see § 24.6.

§ 24.5 Comparison of Residues:

Q: How is Res on $H^1(X, \Omega)$ related to Res on $\mathcal{E}^{(1,0)}(X)$?

A: We can say so on a subspace of $H^1(\mathfrak{U}, \Omega)$, namely the subspace of Mittag-Leffler distributions

(We know by Serre duality that $\dim H^1(\mathfrak{U}, \Omega) = 1$, if $\mathfrak{U}$ is nice enough, eg Leray, so this subspace is either 0 or all of $H^1(X,\Omega)$. The latter will be the case!)

Definition Fix an open covering $\underline{\mathcal{U}}=(U_i)_{i\in I}$ of X & $(g_i)_i \in C^0(\underline{\mathcal{U}}, \mathcal{M}^{(1)})$ with

$(\partial \underline{g})_{ij} = g_j - g_i \in \Omega(U_{ij})$. Then $\partial \underline{g} \in Z^1(\underline{\mathcal{U}}, \Omega)$

We call $(g_i)_i$ a Mittag-Leffler distribution for $\underline{\mathcal{U}}$

Example: $X=\mathbb{P}^1$ $\underline{\mathcal{U}}=(\mathcal{U}_0, \mathcal{U}_\infty)$ $\Rightarrow$ $\underline{g}=(-\frac{dz}{z}, 0)$ is a ML distrib.

$\partial \underline{g} = \frac{dz}{z} \in \Omega(\mathbb{C}^*)$.

Key: We have a nice formula for $\mathrm{Res}([\partial \underline{g}])$ if $(g_i)_i$ is a ML distrib

Theorem 2: $\mathrm{Res}([\partial \underline{g}]) = \sum_{a\in X} \mathrm{Res}_a(g_{i(a)}) =: \mathrm{Res}(\underline{g})$ if $a\in U_{i(a)}$.

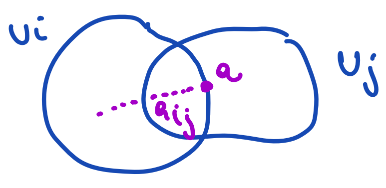
Note: If $a \in U_i \cap U_j$, then $\mathrm{Res}_a(g_i) = \mathrm{Res}_a(g_j)$ because $g_i - g_j \in \Omega(U_{ij})$

This says the choice of $i(a)$ is irrelevant.

Lemma Given g_i : $\mathrm{Res}_a(g_i) \neq 0$ for only finitely many $a\in X$

In particular since X is compact, we can replace $\underline{\mathcal{U}}$ by a finite subcover. This will show the sum in (RHS) is finite.

Proof By construction, we know the poles of g_i's are discrete & cannot accumulate in U_i. If $\{$poles of $g_i\}$ were infinite, then find a subsequence $(a_{ij})_j^{\in U_i}$ converging to $a \in \partial U_i \subseteq X$. ($X$ is locally Euclidean & compact) But $a\in U_j$ will say that g_j will have a as a pole & that a is an accumulation pt of poles of g_j Contr!.



Proof of Theorem 2: To show $\mathrm{Res}([\partial \underline{g}]) = \mathrm{Res}(\underline{g})$ we need to find $\omega \in \mathcal{E}^{(2)}(X)$

with $\delta\omega = [\partial \underline{g}]$ on $H^1(X,\Omega)$.

$(\partial g)_{ij} = g_j - g_i \in \Omega(U_{ij}) \subseteq \mathcal{E}^{(1,0)}(U_{ij})$ & $H^1(X, \mathcal{E}^{(1,0)}) = 0$, so

$\exists\ \underline{\eta} \in C^0(\underline{U}, \mathcal{E}^{(1,0)})$ with $(\partial g)_{ij} = (\partial \underline{\eta})_{ij}$, ie $g_j - g_i = \eta_j - \eta_i$ on U_{ij}.

<u>Claim 1:</u> $d(g_j - g_i) = d''(g_j - g_i) = 0$ $\quad (d''|_{\Omega} = 0)$

Pf/ • Locally on a chart: $g_j = f_j\, dz$ $\quad f_j \in \mathcal{E}$

$\Rightarrow dg_j := df_j \wedge dz = \left(\frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar z} d\bar z\right) \wedge dz = -\frac{\partial f}{\partial \bar z}\, dz \wedge d\bar z$

$d'' g_j := d'' f_j \wedge dz = \left(\frac{\partial f}{\partial \bar z} d\bar z\right) \wedge dz = -\frac{\partial f}{\partial \bar z}\, dz \wedge d\bar z$ // ✓

• $g_j - g_i$ is holomorphic 1-form so $f_j - f_i \in \mathcal{O}(U_{ij})$ ie $d''(f_j - f_i) = 0$

$\Longrightarrow d(\eta_j - \eta_i) = 0$ ie $d\eta_j = d\eta_i$ on U_{ij}

Since $\mathcal{E}^{(2)}$ is a sheaf, the sections $d\eta_i \in \mathcal{E}^{(2)}(U_i)$ glue to a ! $\omega \in \mathcal{E}^{(2)}(X)$ with $\omega|_{U_i} = d\eta_i$ $\forall i$.

<u>Claim 2:</u> $\delta^* \omega = \partial g$ on $Z^1(\underline{U}, \Omega)$ by construction + def of δ^*

$\Longrightarrow \operatorname{Res}([\partial g]) = \frac{1}{2\pi i} \iint_X \omega$

To finish, we need to show (RHS) $= \operatorname{Res}(g)$

Write $\{a_1, \ldots, a_n\}$ for the set of poles of g $\left(= \bigcup_{i \in I} \text{poles}(g_i) \right)$

Set $X' = X \smallsetminus \{a_1, \ldots, a_n\}$.

$\eta_j - g_j = \eta_i - g_i$ on $U_{ij} \cap X'$ $\Longrightarrow$ [sheaf axiom] $\exists\ \sigma \in \mathcal{E}^{(1,0)}(X')$ with $\sigma|_{U_i} = \eta_i - g_i$

$\Longrightarrow \boxed{\omega = d(\sigma)}$ on X' (because $g_i \in \Omega(U_i \cap X')$ so $dg_i = 0$ $\forall i$)

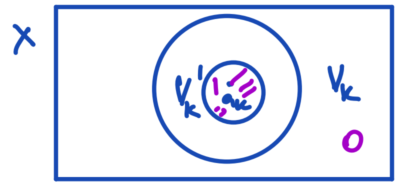
<u>GOAL:</u> Extend (RHS) to X so that residues at a_k's appear naturally:

• Given a_k, pick $i_k \in I$ with $a_k \in U_{i_k}$.

• Fix pairwise disjoint coord charts (V_k, z_k) around each a_k with $V_k \subseteq U_{i_k}$, $V_k \underset{z_k}{\simeq} \mathbb{D}$

In particular, $a_j \notin V_k$ if $j \neq k$.

- Build bump functions $f_k \in \mathcal{E}(X)$ with
  (1) $\mathrm{Supp}(f_k) \subseteq V_k$
  (2) $f_k = 1$ locally around a_k (say on open $V_k' \subset V_k$)



Define $\boxed{\psi := 1 - (f_1 + \cdots + f_n)} \in \mathcal{E}(X) \quad \Rightarrow \psi|_{V_k'} = 0 \quad \forall \sigma$

$\Rightarrow \psi\sigma \in \mathcal{E}^{(1,0)}(X)$ by defining it as 0 on $\bigcup_{k=1}^{n} V_k'$.

By Stokes' Thm, $\iint_X d(\psi\sigma) = 0$.

<u>Claim 3:</u> $d(f_k\sigma) \in \mathcal{E}^{(2)}(X')$ extends to X

Pf/ • $d(f_k\sigma)|_{V_j'} = d(f_k\sigma|_{V_j'}) = d(0) = 0$ for $j \neq k$. so the extension to a_j is trivial.

• On $V_k' \setminus \{a_k\}$, we have $d(f_k\sigma) = d(\sigma) = d(\sigma|_{U_i}) = d(\eta_i - g_i) = d(\eta_i)$
$\quad\quad g_i \in \Omega(V_k')|_{V_k'}$

Since $\eta_i \in \mathcal{E}^{(1,0)}(U_i)$ & $d\eta_i \in \mathcal{E}^{(2)}(U_i)$ then $d(f_k\sigma)$ extends to a_k. ∎

$$\Rightarrow \omega = d(\psi\sigma) + \sum_{k=1}^{n} d(f_k\sigma)$$

$$\Rightarrow \iint_X \omega = \underbrace{\iint_X d(\psi\sigma)}_{=0} + \sum_{k=1}^{n} \iint_X d(f_k\sigma) \underset{\mathrm{supp} f_k \subseteq V_k}{=} \sum_{k=1}^{n} \iint_{V_k} d(f_k\sigma)$$

$$= \sum_{k=1}^{n} \iint_{V_k} d(f_k\eta_{i(k)} - f_k g_{i(k)}) = \sum_{k=1}^{n} \underbrace{\iint_{V_k} d(f_k\eta_{i(k)})}_{= \iint_X d(f_k\eta_{i(k)}) = 0 \text{ by Stokes' Thm}} - \sum_{k=1}^{n} \iint_{V_k} d(f_k g_{i(k)})$$

<u>Claim:</u> $\iint_{V_k} d(f_k g_{i(k)}) = -2\pi i \,\mathrm{Res}_{a_k}(g_{i(k)})$.

Pf/ $$\iint_{V_k} d(f_k g_{i(k)}) \overset{\mathrm{Supp} f_k \subseteq V_k}{=} \lim_{\varepsilon\to 0^+} \iint_{\varepsilon \leq |z-a_k| \leq R} d(f_k g_{ik}) = \lim_{\varepsilon \to 0^+} \underbrace{\int_{|z|=R} f_k g_i}_{=0 \text{ if } R \gg 0} - \int_{|z|=\varepsilon} f_k g_i$$

$$\underset{R \gg 0}{=} -\lim_{\varepsilon\to 0^+} \int_{|z|=\varepsilon} g_i = -2\pi i \,\mathrm{Res}_{a_k}(g_i)$$ ∎

Conclude: $\frac{1}{2\pi i}\iint_X \omega = \sum_{k=1}^{n} \mathrm{Res}_{a_k}(g_{(k)}) = \mathrm{Res}(g)$ as we wanted.

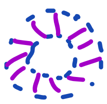
Proposition: $\exists$ a Mittag-Leffler distribution $\underline{g} \in C^0(X, \mathcal{M}^{(1)})$ with $\mathrm{Res}(\underline{g}) = 1$

Therefore, $[\partial g] \neq 0$ in $H^1(X, \Omega)$. & $\mathrm{Res}: H^1(\mathbb{C}, \Omega) \to \mathbb{C}$ is not $\underset{\equiv 0}{\text{const}}$.

Proof: Given $a \in X$, pick (U, z) chart around a, $\underset{a \mapsto 0}{U \overset{z}{\cong} \mathbb{D}}$.

& consider the covering $\underline{\mathcal{U}} = \{ U_0 := U, U_1 := X \setminus z^{-1}(\overline{\mathbb{D}}_{1/2}) \}$

Set $g = (g_0, g_1)$ $\quad g_0 = \frac{dz}{z}$ & $g_1 \equiv 0$.

- On $U_0 \cap U_1 =$  $\frac{dz}{z} \in \Omega(U_{01})$ so $\underline{g}$ is a Mittag-Leffler distrib.
- $\mathrm{Res}(g) = \mathrm{Res}_0(g_0) = 1$

$\square$

§ 24.6 Proof of Residue Theorem:

For each a_j, we pick a coordinate nbhd (U_j, z_j) so that

(1) $\{U_j\}_{j=1}^{n}$ are pairwise disjoint

(2) $U_j \xrightarrow{z_j} \mathbb{D}$ with $z_j(a_j) = 0$.

Next, we build bump functions $f_1, \dots, f_n$ interpolating between U_j & an open $V_j \subseteq U_j$

Conditions: (1) $f_j : X \longrightarrow [0,1]$ is smooth

(2) $\exists\, a_j \in V_j \subseteq U_j$ open with $f_j|_{V_j} = 1$, $f_j = 0$ outside U_j &

$\mathrm{supp}\, f_j \subset U_j$ is compact.

Define $g = 1 - (f_1 + \dots + f_n)$ Then, g is smooth on X & $g|_{V_j} = 0$ $\forall j$.

In particular $g\omega \in \mathcal{E}^{(1)}(X)$ is holomorphic (assign it value 0 on a_j $\forall j$)

& has compact support.

By Stokes' Theorem $\iint_X d(g\omega) = 0$

Then $\iint_X d\omega = \sum_{j=1}^{n} \iint_X d(f_j\omega)$.

<u>Claim 1:</u> $\iint_X d\omega = 0$

Pf/ Write $X' = X \setminus \bigcup V_j$. Then $\omega|_{X'} \in \mathcal{E}^{(1)}(X')$ is holomorphic

so $\iint_{X'} d\omega = 0$ by Thm 2. $\Rightarrow \iint_X d\omega = \sum_{j=1}^{n} \iint_{V_j} d\omega$

Since V_j can be arbitrarily small, the (RHS) has limit 0 as $V_j \longrightarrow a_j$.

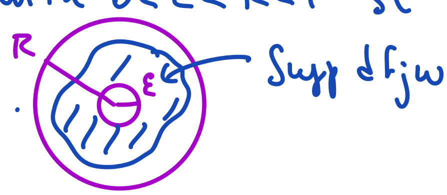
<u>Claim 2:</u> $\iint_X d(f_j\omega) = -2\pi i \, \mathrm{Res}_{a_j}\omega$

Pf/ Since Supp $df_j\omega \subseteq U_j$, we have $\iint_X df_j\omega = \iint_{U_j} df_j\omega$

We identify U_j with $\mathbb{D}$ via $z_j : U_j \longrightarrow \mathbb{D}$ & assume $f_j : \mathbb{D} \longrightarrow \mathbb{R}$, ω is a 1-form on $\mathbb{D}$

By compactness of Supp $df_j\omega$ & $0 \in V_j$ open, we can find ε, R with $0<\varepsilon<R<1$ st

(1) Supp $f_j \subseteq \{|z_j|<R\}$ & (2) $f_j|_{\{|z_j|<\varepsilon\}} = 1$.



Then $$\iint_{U_j} df_j\omega = \iint_{\substack{\varepsilon\le|z_j|\le R \\ \text{(Annulus)}}} df_j\omega = \overbrace{\int_{|z_j|=R} f_K\omega}^{=0} - \int_{|z_j|=\varepsilon} f_K\omega = -\int_{|z_j|=\varepsilon}\omega$$

$= -2\pi i \, \mathrm{Res}_0\omega$ by the Residue Thm in the complex plane.

Using the Residue theorem, we can recover Corollary 2 §5.3

Corollary: If X is a compact RS & $f: X \longrightarrow \mathbb{P}^1$ is holomorphic non-constant, then # zeroes of f = # poles of f (counted with mult)

By shifting, we see that the size of each fiber of f is constant (f is surj.)

Proof Consider the 1-form $\omega = \frac{df}{f}$ it is meromorphic on X & its only singularities are the zeroes and poles of f.

$\mathrm{Res}_a \omega$ = order of vanishing as a zero / − order of pole. for each singularity a of ω

Since $\sum_{j=1}^{n} \mathrm{Res}_{a_j} \omega = 0$ where $\{a_1, \dots, a_n\} = f^{-1}(0) \cup f^{-1}(\infty)$ by Theorem,

separating this sum by sign will give us the statement. $\square$