Lecture XXV: Riemann-Hurwitz & Topology of compact R.S.

Recall Given $\omega \in \mathcal{M}^{(1)}(X) \setminus \{0\}$ on X compact Riemann surface, then $K = (\omega)$ is call the canonical divisor of X. It's unique up to linear equivalence

$$\deg K = \sum_{x \in X} K(x) = 2g-2 \qquad \text{where } g = \text{genus}(X) = \dim H^1(X, \mathcal{O})$$
$$= \dim H^0(X, \Omega)$$
↳ Serre duality.

§25.1 The Riemann-Hurwitz Theorem:

Fix X, Y compact R.S & $f: X \to Y$ non-constant holomorphic map of degree n

$$\Rightarrow n = |f^{-1}(y)| = \sum_{x \in f^{-1}(y)}' \nu(f,x) \qquad \text{where } \nu(f,x) \text{ is the multiplicity of}$$

f at x (Locally near x & y we write $f: \mathbb{D}_0 \longrightarrow \mathbb{D}_0$ as $f(z) = z^k$, $k = \nu(f,x)$)

Definition. $b(f,x) = \nu(f,x) - 1$ is called the <u>branching order</u> of f at x

In particular: f unbranched at $x \iff b(f,x) = 0 \iff \nu(f,x) = 1 \iff f$ local homeomorphism near x.

Lemma: (1) $D = \sum_{x \in X} b(f,x)[x]$ is a divisor on X (branching divisor of f)

(2) $D' = \sum_{\substack{y \in Y \\ y \in f(\operatorname{Supp} D)}} [y]$ is a divisor on Y (critical values of f)

Proof (1) We know $\operatorname{Supp} D$ is discrete & closed because f is a proper, non-constant holo map (§5.3)
X is compact, so $\operatorname{Supp} D$ is discrete & compact, ie finite

Definition. $b(f) := \sum_{x \in X}' b(f,x) = \deg D$ is called the <u>total branching order</u> of f.

Theorem: (Riemann-Hurwitz Formula) In the above setting, we have

$$g_X = \frac{b(f)}{2} + \text{degree}(f)\,(g_Y - 1) + 1$$

Equivalently, $2g_X - 2 = b(f) + \text{degree}(f)\,(2g_Y - 2)$.

Proof: We consider canonical divisors on X & Y.

For Y: Pick any $\omega_Y \in \mathcal{M}^{(1)}(Y)\setminus\{0\}$ & set $K_Y = (\omega_Y)$

For X: Use $f^*\omega_Y \in \mathcal{M}^{(1)}(X)\setminus\{0\}$ & set $K_X = (f^*\omega_Y)$

• Next, we write $f^*\omega_X$ in local coordinates to compute K_X

• How? Fix $x \in X$ & $y = f(x) \in Y$ & pick local coordinates (U, z) for x with (V, z') for y

$$\begin{tikzcd} x \arrow[r, mapsto] & 0 & \\ U \arrow[r, "\sim"', "z"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{D} \arrow[d] & z \arrow[d, mapsto] \\ V \arrow[r, "\sim"', "z'"] & \mathbb{D} & z' = z^k \\ y \arrow[r, mapsto] & 0 & \end{tikzcd} \qquad k = \nu(f,x)$$

$\Rightarrow$ If $\omega_Y = \varphi(z')\, dz' \in \mathcal{M}^{(1)}(V)$ locally around y, we have

$$f^*\omega_Y = \varphi(z'\circ f)\, d(z'\circ f) = \varphi(z^k)\, dz^k = \varphi(z^k)\, k z^{k-1} dz \in \mathcal{M}^{(1)}(U)$$

$$\Rightarrow \operatorname{ord}_x(f^*\omega_Y) = k-1 + \operatorname{ord}_0 \varphi(z^k) = k-1 + k\cdot \operatorname{ord}_0 \varphi = \underbrace{k-1}_{=b(f,x)} + \underbrace{k}_{\nu(f,x)} \operatorname{ord}_y \omega_Y .$$

• Summing over $x \in f^{-1}(y)$ we get:

$$\sum_{x\in f^{-1}(y)} \operatorname{ord}_x(f^*\omega_Y) = \sum_{x\in f^{-1}(y)} b(f,x) + \underbrace{\sum_{x\in f^{-1}(y)} \nu(f,y)}_{=\text{degree}(f)} \cdot \operatorname{ord}_y \omega_Y$$

Now, we sum over y & we see (LHS) is finite $(= \deg(f^*\omega_Y))$

$$\deg f^*\omega_Y = \sum_{x\in X} \operatorname{ord}_x(f^*\omega_Y) = \sum_{y\in Y}\sum_{x\in f^{-1}(y)} \operatorname{ord}_x(f^*\omega_Y)$$

$$= \underbrace{\sum_{y\in Y}\sum_{x\in f^{-1}(y)} b(f,x)}_{=b(f)} + \text{degree}(f) \underbrace{\sum_{y\in Y} \operatorname{ord}_y(\omega_Y)}_{=\deg(\omega_Y)}$$

$$\Rightarrow 2g_X - 2 = b(f) + \text{degree}(f)\,(2g_Y - 2) \quad \checkmark$$

Example: $Y = \mathbb{P}^1$ & $f: X \longrightarrow \mathbb{P}^1$ is a degree n holomorphic non-const map with X compact,

we get $2g - 2 = b(f) + n\,(2\cdot 0 - 2)$ In particular, $b(f)$ is even!

$$\boxed{g = \frac{b(f)}{2} - n + 1}$$

Special case: $n=2$, we say the map $f: X \longrightarrow \mathbb{P}^1$ is a hyperelliptic cover (more on this later!)

§ 25.2 Topological classification & Euler characteristic

Next, we want to study compact R.S. from the topological perspective. This will give another proof of Riemann-Hurwitz.

Our starting point is a classification theorem of orientable closed surfaces of genus g:

Classification Thm: Fix S an orientable differentiable surface of genus g. Then,

$$S \simeq \Sigma_g = \Sigma_{g-1} \# \mathbb{T}^2 = \underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{g \text{ times}}$$

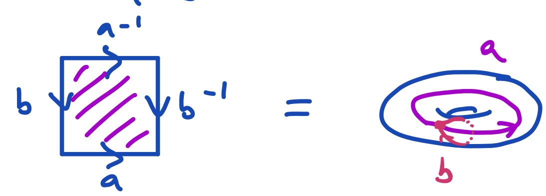
where $\mathbb{T}^2$ is the 2-torus $= \mathbb{S}^1 \times \mathbb{S}^1$ & $\#$ denotes connected sum

Eg

So Σ_g :

(= a sphere $\mathbb{S}^2$ with g handles attached) $\quad \overset{\shortparallel}{\mathbb{S}^2 \setminus g \text{ discs}} \quad g \text{ handles} \setminus \mathbb{D}$.

Examples: $g=0$ $\quad \mathbb{S}^2$ $\quad$, $g=1$ $\quad \mathbb{S}^1 \times \mathbb{S}^1$



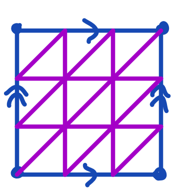
- These 2 can be obtained by gluing edges of a polygon & can be triangulated using these polygons:

$g=0$ $\quad \mathbb{S}^2$

3 vertices
3 edges
2 Δ (front & back)

$\chi(\mathbb{S}^2) = 3-3+2 = 2 = 2-2g$

$g=1$ $\quad \mathbb{S}^1 \times \mathbb{S}^1$



$9 = 18 \cdot 3/6$ vertices (each vertex is in 6 triangles)

$27 = 18 \cdot 3/2$ edges (each edge is shared by 2 triangles)

18 triangles

$\chi(\mathbb{S}^1 \times \mathbb{S}^1) = 9-27+18 = 0 = 2-2g$

Definition: The Euler characteristic of a smooth orientable surface without boundary is

$$\chi(S) = rk(H^0(X,\mathbb{Z})) - rk(H^1(X,\mathbb{Z})) + rk(H^2(X,\mathbb{Z}))$$

Obs: If S has genus g, then $\chi(S) = 1 - \underbrace{2g}_{\text{2 cycles per handle}} + 1 = 2-2g$

Proposition: Fix a compact 2-manifold S (possible with boundary) & assume S has a triangulation Δ. Then $\chi_{(\Delta,S)} := V - E + T$ with $V = \#$ vertices of Δ, $E = \#$ edges of Δ, $T = \#$ triangles of Δ

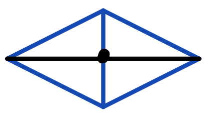
is invariant under refinements of Δ.

Furthermore, if S is orientable without boundary, then S can be triangulated & $\chi(S) = \chi_{(\Delta,S)}$.

Proof: 1 Need to see (RHS) is invariant under refinements of triangulations

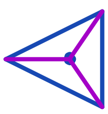
3 ways of refining:

1) Subdivide an edge by adding a vertex $\Rightarrow$ need to subdivide the 2 triangles containing it



$$\begin{aligned} V' &= V+1 \\ E' &= E+3 \\ T' &= T+2 \end{aligned} \Rightarrow V'-E'+T' = V-E+T$$

2) Add a vertex in the interior of a triangle $\Rightarrow$ stellar subdivision of the Δ



$$\begin{aligned} V' &= V+1 \\ E' &= E+3 \\ T' &= T+2 \end{aligned} \Rightarrow V'-E'+T' = V-E+T$$

These 2 operations are called elementary refinements of triangulations for surfaces without boundary

3) If S has a boundary & we have an edge $e \subseteq \partial S$, we can add 1 more refinement operation: subdivide an edge e in ∂S by adding a pt. Then subdivide the ! triangle containing e: e

$$\begin{aligned} V' &= V+1 \\ E' &= E+2 \\ T' &= T+1 \end{aligned} \Rightarrow V'-E'+T' = V-E+T$$

Key: Any 2 triangulations have a common refinement (superimpose them & add vertices & edges that turn this decomposition into a triangulation).

② To prove part II, we need to show we can triangulate Σ_g & get $\chi(\Sigma_g) = \chi(\Sigma_g, \Delta)$

We can do this by induction on g.

- <u>Base cases</u> $g=0$ & 1 was done in examples above.
- <u>Inductive step</u>: Use $\Sigma_{g+1} = \Sigma_g \# \mathbb{T}^2$. We can take out the interior of any triangle in the triangulation of Σ_g & $\Sigma_1 = \mathbb{T}^2_\Delta$ glue them.

The new triangulation will have:

$$V = V_1 + V_2 - 3 \quad (\text{we identified vertices on 2 triangles, one on each side})$$

$$E = E_1 + E_2 - 3 \quad (\text{——— edges ———})$$

$$T = T_1 - 1 + T_2 - 1 \quad (\text{we removed 1 triangle on each side})$$

$$\Rightarrow V - E + T = \underbrace{(V_1 - E_1 + T_1)}_{\text{IH}} + \underbrace{(V_2 - E_2 + T_2)}_{g=1 \text{ case}} + (-3 - (-3) - 2)$$

$$= 2 - 2g + 0 + (-2)$$

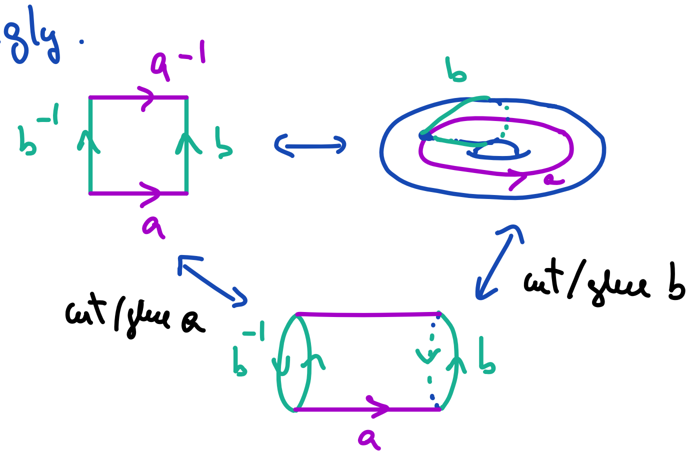
$$= 2g = 2 - 2(g+1) = \chi(S) \qquad \square$$

Q: How can we compute $\chi(S)$ for S compact R.S?

A: Use polygon description of Σ_g given by the following topological model for Σ_g.

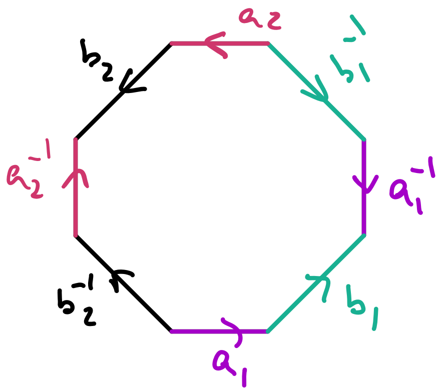
<u>Theorem</u>: A genus $g>0$ compact R.S can be obtained by taking a $4g$-gon oriented counterclockwise, with edges $a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}, \dots, a_g, b_g, a_g^{-1}, b_g^{-1}$ identified accordingly.

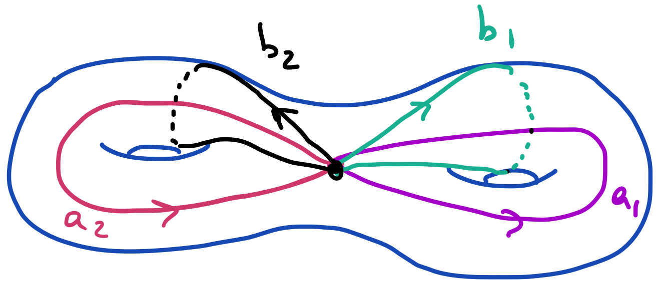
<u>Example</u>: $g=1$ (4-gon)



cut the Torus along these 2 loops

$g=2$
(8 gon)

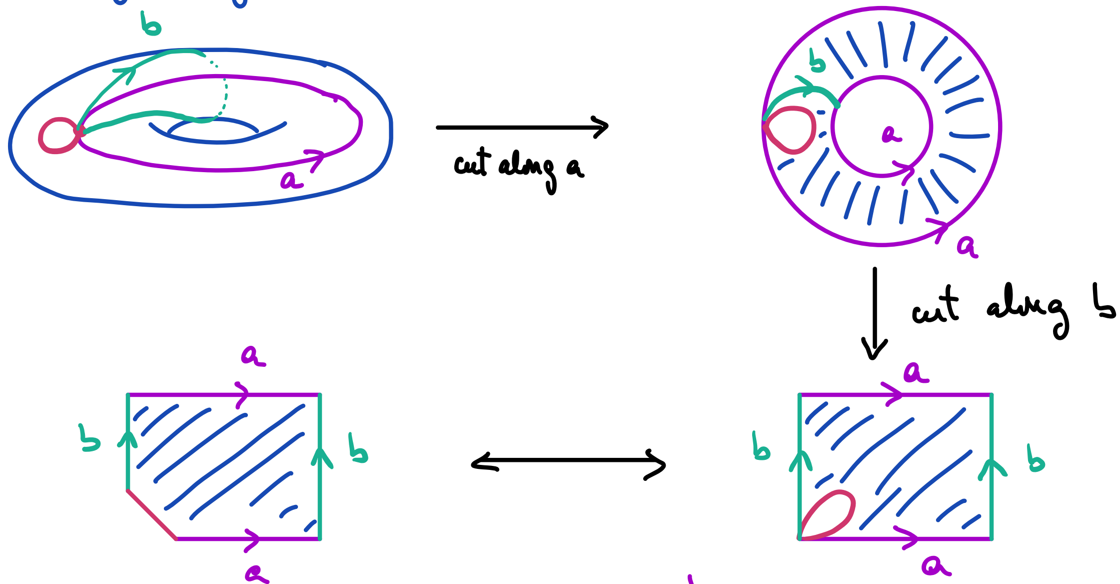




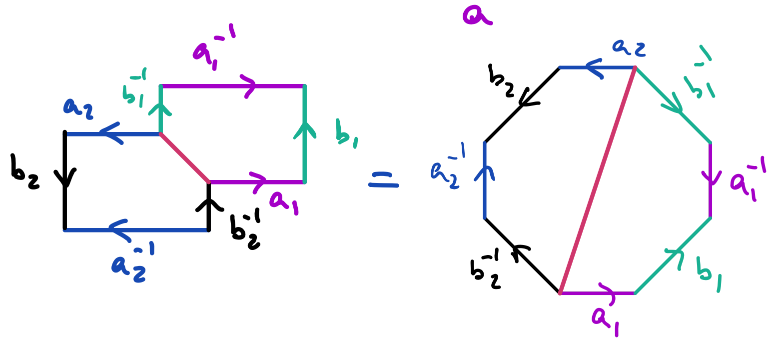
How to interpret this? Glue two polygons corresponding to $\mathbb{T}^2 \setminus \mathbb{D}$.

(In general: $\Sigma_g = \Sigma_{g-1} \# \mathbb{T}^2$)

$\mathbb{T}^2 \setminus \mathbb{D}$:



$\Rightarrow$ $\mathbb{T}^2 \# \mathbb{T}^2$ is obtained as



- Induction on g gives the general statement.

Using this topological model, we can reprove the Riemann-Hurwitz formula.

<u>Proof of Riemann-Hurwitz</u>

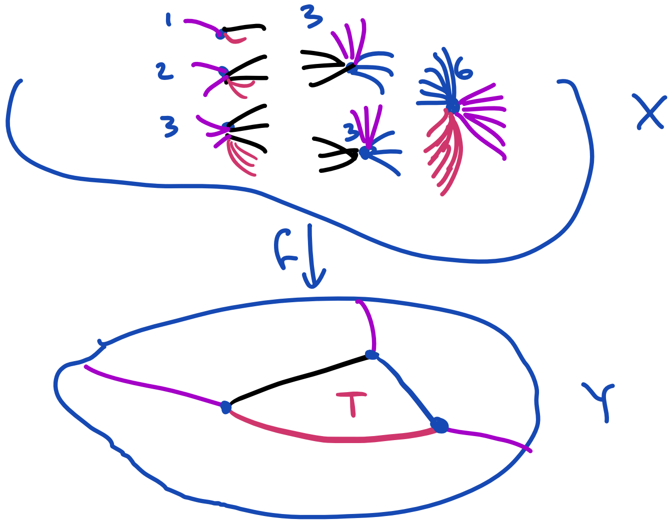
We write a triangulation for Y & use f to lift it to a triangulation of X

We pick a triangulation of Y whose vertices include all critical points of f.

How can we build a triangulation of X from this?

Pictorially:

degree $f = 6$



local multiplicities $1, 2, 3, 6$

B = critical values of f = f(Branch divisor)

- $f|_{X\setminus f^{-1}(B)} : X\setminus f^{-1}(B) \longrightarrow Y\setminus B$ is an n-sheeted covering

$\Rightarrow$ If $\Delta \subseteq Y$ is a triangle, then $f^{-1}(\mathrm{Int}(\Delta)) \simeq \bigsqcup_n \mathrm{Int}(\Delta)$

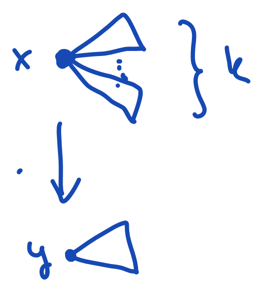
$\Rightarrow$ Taking closure of $\bigsqcup_{\Delta\subseteq Y} f^{-1}(\mathrm{Int}(\Delta))$ we get a triangulation of X induced by f

- By refining the input triangulation of Y (by stellar subdivisions), we may assume each Δ in Y has at most 1 critical pt as a vertex.

- If $\Delta \subseteq Y\setminus B$, then Δ lifts to n triangles

$\Rightarrow$ Vertices, Edges of Δ are also multiplied by n

- If Δ has a vertex y in B & $x \in f^{-1}(y)$ is a branch pt with $\nu(f,x) = k > 1$, then, the map f around x behaves like $z \mapsto z^k$.



Then, the preimage of T around x consists of k many Δ, all with a common vertex x.

$\Rightarrow$ Number of triangles & edges in the preimage is $\sum'_{x\in f^{-1}(y)} \nu(f,x) = n$

______ vertices in the preimage of a critical value y is $\sum_{x\in f^{-1}(y)} 1$

But $$\sum'_{x\in f^{-1}(y)} 1 = \sum'_{x\in f^{-1}(y)} \Big(\nu(f,x) - \underbrace{(\nu(f,x)-1)}_{=\, b(f,x)}\Big) = n - \sum_{x\in f^{-1}(y)} b(f,x)$$

$$\Rightarrow \chi(X) = V_X - E_X + T_X \quad \text{by Proposition}$$

$$= n(V_Y - |B|) + \sum_{y \in B} \sum_{x \in f^{-1}(y)} 1 - nE_Y + nT_Y$$

separate count for vertices in B or not

$$= n(V_Y - |B|) + \sum_{y \in B} \left(n - \sum_{x \in f^{-1}(y)} b(f,x)\right) - nE_Y + nT_Y$$

$$= n(V_Y - E_Y + T_Y) - b(f) \underset{\text{Proposition}}{=} \operatorname{degree}(f)\,\chi(Y) - b(f)$$

Changing signs we get $\quad 2g_X - 2 = -\chi(X) = b(f) + \operatorname{degree}(f)(2g_Y - 2)$

$\square$