Lecture XXVI: Hyperelliptic curves, degree & genus of plane curves, linear systems

Recall Given $\omega \in \mathcal{M}^{(1)}(X) \setminus \{0\}$ on X compact Riemann surface, then $K = (\omega)$ is call the canonical divisor of X. It's unique up to linear equivalence

$$\deg K = \sum_{x \in X} K(x) = 2g-2 \qquad \text{where } g = \text{genus}(X) = \dim H^1(X, \mathcal{O})$$
$$= \dim H^0(X, \Omega)$$
$\hookrightarrow$ Serre duality.

Theorem: (Riemann-Hurwitz Formula) Given X, Y compact R.S, $f: X \longrightarrow Y$ proper, holomorphic non-constant, we have $g_X = \frac{b(f)}{2} + \text{degree}(f)(g_Y - 1) + 1$

Equivalently, $\underbrace{2g_X - 2}_{= -\chi(X)} = b(f) + \text{degree}(f) \, (\underbrace{2g_Y - 2}_{= -\chi(Y)})$.

Here, $b(f) = \sum_{x \in X}' \nu(f,x)$ (total branching order) $\qquad \nu(f,x) =$ local mult of f at x $\left(\begin{array}{c} z \longmapsto z^{\nu(f,x)} \\ \text{locally near } x. \end{array} \right)$

Example: $Y = \mathbb{P}^1$ & $f: X \longrightarrow \mathbb{P}^1$ is a degree n holomorphic non-const map with X compact, we get $2g - 2 = b(f) + n\,(2 \cdot 0 - 2)$ In particular, $b(f)$ is even!

$$\boxed{g = \frac{b(f)}{2} - n + 1}$$

Special case: $n = 2$, we say the map $f: X \longrightarrow \mathbb{P}^1$ is a hyperelliptic cover

§26.1 Hyperelliptic covers of $\mathbb{P}^1$ & holomorphic 1-forms:

Prototypical example: $X \longrightarrow \mathbb{P}^1$ where X is the R.S. build as the algebraic function (X, p, f) defined by $Q = T^2 - h(z) \in \mathbb{C}(z)[T]$ where $h(z) = (z - \alpha_1) \cdots (z - \alpha_n)$ $\quad n \geq 2 \quad$ Write $f = \sqrt{h(z)}$

- We discussed these in Lecture 13 & saw that $X \simeq \Sigma_g$ where $g = \left\lceil \frac{n}{2} \right\rceil - 1 = \left\lfloor \frac{n-1}{2} \right\rfloor$

Map $f: (T, z) \longmapsto T$ in $\mathcal{M}(X)$

Unbranched away from 0 & ∞ $\begin{cases} \text{If } n \text{ even}: & \infty \text{ is } \underline{\text{not}} \text{ a critical pt} \\ \text{If } n \text{ odd}: & \infty \text{ is a critical pt.} \end{cases}$

Note $b(f)$ is even by Example above & $b(f) = \#$ branch pts $= \begin{cases} n & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd} \end{cases}$

By Serre duality: $\dim H^0(X, \Omega) = \dim H^1(X, \mathcal{O})^\vee = g$.

Q: Can we find a basis of $\Omega(X)$ using $f \in \mathcal{M}(X)$? A: YES

Proposition: $\left\{ \eta_j := \dfrac{z^{j-1}}{\sqrt{h(z)}}dz = \dfrac{z^{j-1}dz}{f} \quad j = 1, \dots, \left\lfloor \dfrac{n-1}{2} \right\rfloor \right\}$ is a basis for $\Omega(X)$

Proof: Recall the construction of X. We have $X' \subset X$ & two maps

$$\begin{array}{ccc} X' & \xrightarrow[\substack{2:1 \\ \text{unbranched cover}}]{T} & \mathbb{C} \\ {\scriptstyle \text{holo } z} \big\downarrow & & \\ \mathbb{C} \setminus \{\alpha_1, \dots, \alpha_n\} & & \end{array}$$

Set $X := X' \cup \{p_1 \dots p_n\} \cup \begin{cases} \{\infty\} & n \text{ odd} \\ \{\infty_+, \infty_-\} & n \text{ even} \end{cases}$

- Near p_K: map z locally has the form $t \longmapsto t^2$
$p_j \in \mathbb{D} \ni t$, $t^2 \in \mathbb{D} \ni \alpha_j$

- Near ∞: map z locally has the form $t \longmapsto t$ if n odd
$t \longmapsto t^2$ " n even

$\rightsquigarrow$ we get

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{P}^1 \\ {\scriptstyle p} \big\downarrow & & \\ \mathbb{P}^1 & & \end{array}$$

f meromorphic extends T
p holomorphic extends z

Note: $\eta_j = \dfrac{z^{j-1}}{f} dz \in \mathcal{M}(X)$. Q: Why is it holomorphic?

- Away from $p_1, \dots, p_n, 0$ is holomorphic (f doesn't vanish & z is holomorphic)
$\neq 0$
- At $z=0$, we need $\boxed{j-1 \geq 0}$ (assuming $\alpha_K \neq 0$)
- Near p_K:

$$\begin{array}{c} \cdot p_K \\ \big\downarrow \\ \cdot \alpha_k \end{array} \qquad z = \alpha_K + t^2$$

$$\Rightarrow h(z) = t^2 \underbrace{\prod_{\substack{i=1 \\ i \neq K}}^{n} (t^2 + \alpha_K - \alpha_i)}_{\neq 0 \text{ near } t=0 \text{ \& holo}}$$

Locally, we can take $\sqrt{h_{(z)}}$ & get 2 values. This means that

$$T = t\sqrt{\prod_{i\neq k}(t^2-\alpha_j-\alpha_i)} = t g_k(t) \text{ \& } g_k \text{ is holomorphic near } t=0$$

$\Rightarrow$ Near p_k we have: $\eta_j := \frac{z^{j-1}}{f}dz = \frac{(\alpha_k+t^2)^{j-1}\cdot 2t\,dt}{t\,g_k(t)} = \underbrace{\frac{2(\alpha_k+t^2)^{j-1}}{g_k(t)}}_{\in \mathcal{O}_{X,p_k}} dt$

Note: There are no restrictions on j (other than $j-1\geq 0$ if $\alpha_k=0$)

- Near ∞: We have to analyze n odd / even separately

$$T^2 = z^n \underbrace{\prod_{j=1}^{n}\left(1-\frac{\alpha_j}{z}\right)}_{\neq 0 \text{ \& holo near } \infty} \quad \text{near } z=\infty.$$

① N ODD: $z \longmapsto \frac{1}{w^2}$

(∞ is branch pt)

$$dz = \frac{-2\,dw}{w^3}$$

$$T^2 = \frac{1}{w^{2n}}\prod_{i=1}^{n}(1-\alpha_i w^2)$$

$$\Rightarrow f = T = \frac{1}{w^n}\underbrace{\sqrt{\prod_{i=1}^{n}(1-\alpha_i w^2)}}_{=g_\infty \text{ holo near } w=0}$$

$$\Rightarrow \eta_j = \frac{-2w^{n-2j-1}}{g_\infty(w)}dw$$

holo near 0 $\iff n-2j-1\geq 0$

$$\left\lfloor\frac{n-1}{2}\right\rfloor = \frac{n-1}{2} \geq j$$

② N EVEN: $z \longmapsto \frac{1}{w}$

(∞ is not a branch pt)

$$dz = \frac{-dw}{w^2}$$

$$T^2 = \frac{1}{w^n}\prod_{i=1}^{n}(1-\alpha_i w^2)$$

$$f = T = \frac{1}{w^{n/2}}\Big(\pm\underbrace{\sqrt{\prod_{i=1}^{n}(1-\alpha_i w^2)}}_{=g_\infty \text{ holo near } w=0}\Big)$$

(sign depends on which ∞ in X we are considering: there are 2 pts over ∞!)

$$\Rightarrow \eta_j = -\frac{w^{-j+1+\frac{n}{2}-2}}{g_\infty(x)}dw = \frac{-w^{\frac{n}{2}-j-1}}{g_\infty}dw$$

holo near 0 $\iff \frac{n}{2}-j-1\geq 0$

$$\frac{n}{2}-1 \geq j$$

$$\iff \left\lfloor\frac{n-1}{2}\right\rfloor \geq j$$

In both cases, we get $\boxed{\left\lfloor\frac{n-1}{2}\right\rfloor \geq j}$

Conclude $\{\eta_1,\dots,\eta_g\}$ is a basis for $H^0(X,\Omega)$. (li because they have $\neq$ orders of vanishing at 0)

§26.2 Degree of a smooth projective plane curve:

Let $C = \{F(x,y,z) = 0\}$ be a smooth plane curve of degree d, meaning that F is a homogeneous polynomial of degree d with

$$(*) \quad \{F = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0\} = \emptyset \text{ in } \mathbb{P}^2 \qquad (C \text{ is non-singular})$$

Note: As a manifold $\mathbb{P}^2$ can be described by an atlas with 3 open sets

$$U_0 = \{[x:y:z] : x \neq 0\}, \quad U_1 = \{[x:y:z] : y \neq 0\}, \quad U_2 = \{[x:y:z] : z \neq 0\}$$

$$\downarrow \text{homeo} \qquad\qquad \downarrow \text{homeo} \qquad\qquad \downarrow \text{homeo}$$

$$\left(\tfrac{y}{x}, \tfrac{z}{x}\right) \in \mathbb{C}^2 \qquad \left(\tfrac{x}{y}, \tfrac{z}{y}\right) \in \mathbb{C}^2 \qquad \left(\tfrac{x}{z}, \tfrac{y}{z}\right) \in \mathbb{C}^2$$

$\Rightarrow C$ is covered by 3 opens
$$C_0 = C \cap U_0 = \{F(1,y,z) = 0\} \subseteq \mathbb{C}^2$$
$$C_1 = C \cap U_1 = \{F(x,1,z) = 0\} \subseteq \mathbb{C}^2$$
$$C_2 = C \cap U_2 = \{F(x,y,1) = 0\} \subseteq \mathbb{C}^2$$

<u>Lemma 1</u>: $(*)$ happens $\iff$ C_0, C_1, C_2 are affine smooth plane curves $\left(f = \frac{\partial f}{\partial u} = \frac{\partial f}{\partial v} = 0\right)$

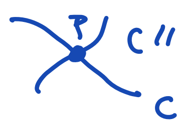
<u>Proof</u>: Euler's formula gives $F = \frac{1}{d} \sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i}$ for any homog. deg d polynomial F in n variables. The result follows from this.

<u>Lemma 2</u>: C is non-singular $\Rightarrow$ F is irreducible

<u>Proof</u>: If $F = G \cdot H$ then $C = C' \cup C''$ $\quad C', C''$ curves
$C' = \{G(x,y,z) = 0\}$
$C'' = \{H(x,y,z) = 0\}$

& the system $\{G = H = 0\} \subseteq \mathbb{P}^2$ has a non-trivial solution p.

The point p will be a solution to $(*)$ Contr!



Q: Why is C a Riemann surface?

A: It has a complex structure, as we now show: We have 3 opens C_0, C_1, C_2

By symmetry, we only need to consider the overlap $C_0 \cap C_1$:

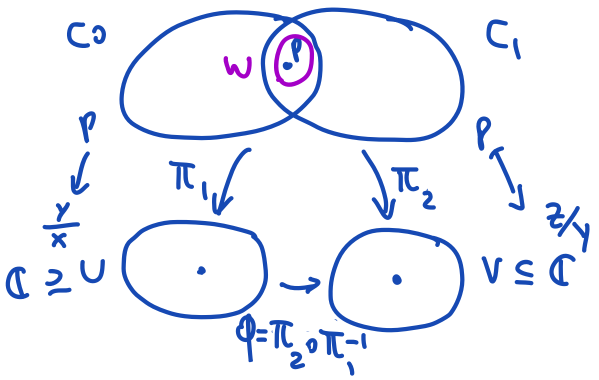
Given $p \in C_0 \cap C_1$, we know $p = [x:y:z]$ with $x \neq 0$, $y \neq 0$.

We can write a chart map p for C using the implicit function theorem on C_0

$\frac{z}{x} = h_2\left(\frac{y}{x}\right)$ & this is a holomorphic function on $W_0 \ni p$ open

Similarly on C_1, we can use the implicit function theorem to write

$\frac{z}{y} = h_1\left(\frac{x}{y}\right)$ as a holomorphic function on $W_1 \ni p$ open



$\phi = \pi_2 \circ \pi_1^{-1}(\alpha) = \pi_2[1, \alpha, h_2(\alpha)]$
$= \frac{h_2(\alpha)}{\alpha}$ which

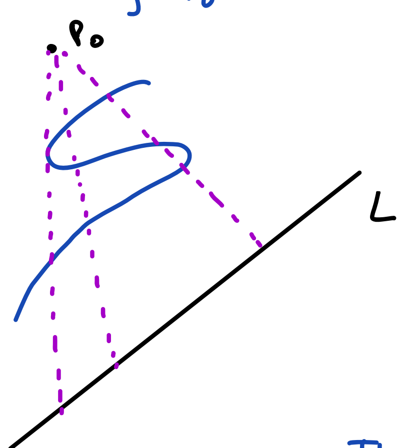
is holomorphic because $0 \notin U$
$(p \in U_1)$

<u>Theorem:</u> The genus & degree of C are related via the formula:

$$g(C) = \frac{(d-1)(d-2)}{2}$$

<u>Proof</u>: We will use a map $C \longrightarrow \mathbb{P}^1$ & the Riemann-Hurwitz formula.

Fix a general pt $p_0 \in \mathbb{P}^2 \setminus C$ & consider the projection from P_0 onto a line $L \subseteq \mathbb{P}^2$ not containing P_0:



$\pi: C \longrightarrow L \simeq \mathbb{P}^1 \subseteq \mathbb{P}^2$
$p \longmapsto \overline{P_0 p} \cap L$

this is a holomorphic non-constant map
(if $\overline{P_0P} \parallel L$, $p \mapsto \infty$.)

<u>Q</u>: What is the degree of this map?

If $Q \in L$ is a pt, then $\pi^{-1}(Q) = \overline{P_0 Q} \cap C$

The points in $\pi^{-1}(Q)$ are of the form $P = P_0 + t\,\overline{P_0 Q}$.

We need to find t solving $F(P) = 0$. This is a univariant polynomial in t of degree d if P_0 & Q are general. <u>Answer:</u> degree d = degree (C)

Moreover: branch pts of π are those pts P in C where $\overline{P_0P}$ is tangent to C at P

Exercise: If P_0 is general, the tangency of this line has multiplicity ≤ 2 ($F(p)=0$ has at most roots of multiplicity 2 $\forall Q \in L$)

Conclude $b(\pi) = \sum_{P\in\pi^{-1}(Q)} \nu(\pi,P)-1 = \#$ pts $P \in \pi^{-1}(Q)$ so that $P_0 \in T_PC$.

Here $T_PC = \{ \frac{\partial F}{\partial x}(p)\, x + \frac{\partial F}{\partial y}(p)\, y + \frac{\partial F}{\partial z}(p)\, z = 0 \}$

If $P_0 = [a_0 : b_0 : c_0]$ then $P_0 \in T_PC \iff \begin{cases} \frac{\partial F}{\partial x}(p)\, a_0 + \frac{\partial F}{\partial y}(p)\, b_0 + \frac{\partial F}{\partial z}(p)\, c_0 = 0 \\ F(p) = 0 \end{cases}$

We have 2 equations in 3 variables ($p=[x,y,z]$), of degrees $d-1$ & d.

Bézout's Theorem says there are $d(d-1)$ solutions. This is $b(\pi)$!

Riemann-Hurwitz formula says: $2g(C)-2 = \deg(f)\,(2g_{(\mathbb{P}^1)}-2) + b(\pi)$.

$$= d(-2) + d(d-1)$$

$\iff 2g(C) = d(d-3)+2 \iff g(C) = \frac{(d-1)(d-2)}{2}$ ✓

§26.3 Linear Systems

Next goals: ① Show that any compact R.S can be embedded as a sm. projective curve in some $\mathbb{P}^r$. We will do this by a linear system associated to a divisor on X (We'll see this can be done if $\deg D = 2g+1$)

② Classify compact R.S of low genera.

Recall $H^0(X, \mathcal{O}_D)$ is a finite dimensional vector space / $\mathbb{C}$ by Riemann-Roch.

$\Rightarrow$ We can consider the projectivization of this space ($\mathbb{P}V = \frac{V \setminus \{0\}}{v \sim \lambda v \text{ for } \lambda\neq 0}$)

Definition: $|D| := \mathbb{P}(H^0(X, \mathcal{O}_D)) \simeq \mathbb{P}^{\dim H^0(X,\mathcal{O}_D)-1}$.

Name: $|D|$ is called the complete linear system associated to D

Definition: A linear system is a linear subspace $\Lambda \subseteq |D|$

We say Λ has degree $d = \deg D$ & dimension $r = \dim \Lambda$
(projective!)

We'll use linear system to write maps $X \longrightarrow \mathbb{P}^r$ & understand when this is an embedding.

Proposition 1: $|D| \simeq \{ E \in \mathrm{Div}(X) : E \geq 0 \text{ \& } E \sim D \}$ as vector spaces

Proof: Given $f \in H^0(X, \mathcal{O}_D)$ $f \neq 0$, we can define an effective divisor
$\mathrm{div}^D(f) := (f) + D \geq 0$ since $f \in \mathcal{O}_D$.
$\mathrm{div}^D(f) \sim D$ since $\mathrm{div}^D(f) - D = (f)$ is principal.

. Moreover: $\mathrm{div}^D(\lambda f) = \mathrm{div}^D(f)$ $\forall \lambda \in \mathbb{C}^\times$ so we have a linear map

$$\Phi \quad \mathbb{P}(H^0(X, \mathcal{O}_D)) \longrightarrow \{ E \in \mathrm{Div}(X) : E \geq 0 \text{ \& } E \sim D \}$$
$$[f] \longmapsto \mathrm{div}^D(f) = (f) + D$$

Claim 1: Φ is injective

Pf/ $\mathrm{div}^D(f_1) = \mathrm{div}^D(f_2) \iff (f_1) = (f_2)$ ie $\mathrm{ord}_x f_1 = \mathrm{ord}_x f_2 \ \forall x$

This ensures that $\frac{f_1}{f_2} \in \mathcal{M}(X)$ satisfies $\mathrm{ord}_x\left(\frac{f_1}{f_2}\right) = 0 \ \forall x$,

so $\frac{f_1}{f_2} \in \mathcal{O}(X)$ since $\mathrm{ord}_x(0) = \infty$. But X is compact so $\frac{f_1}{f_2} = \lambda \in \mathbb{C}$ & $\lambda \neq 0$

Conclude: $f_1 = \lambda f_2$ ie $[f_1] = [f_2]$ in $|D|$.

Claim 2: Φ is surjective

Pf/ Pick $E \geq 0$ $E \sim D$ & pick $f \in \mathcal{M}(X) \setminus \{0\}$ with $E = (f) + D$
then $0 \leq \mathrm{ord}_x E = \mathrm{ord}_x f + \mathrm{ord}_x D$ $\forall x$ forces $\mathrm{ord}_x f \geq -\mathrm{ord}_x D$,
so $f \in \mathcal{O}_D(X)$. **Conclude:** $[f] \in \mathbb{P}(H^0(X, \mathcal{O}_D))$.

Example: ① $X = \mathbb{P}^1$ $\quad D = d[\infty]$ $\quad$ ($\mathrm{Div}(X)/\mathrm{Ppal}(X) \simeq \mathbb{Z}$ by §24.2)

$\Rightarrow |d[\infty]| = \{$all effective divisors of degree d on $\mathbb{P}^1\}$

$\mathcal{O}_{D+K} \simeq \Omega_D$ by Proposition §23.2 $\Rightarrow \mathcal{O}_D \simeq \Omega_{D-K}$

$K = (dz) = -2[\infty] \Rightarrow \mathcal{O}_{d[\infty]} \simeq \Omega_{(d+2)[\infty]}$

$\Rightarrow H^0(\mathbb{P}^1, \mathcal{O}_{d[\infty]}) = H^0(\mathbb{P}^1, \Omega_{(d+2)[\infty]}) = \{0\}$ if $-(d+2) \geq -1$
$-1 \geq d$
by Lemma 1 §24.2

Whereas $H^0(\mathbb{P}^1, \Omega_{(d+2)[\infty]})$ has dim $= d+1$ if $-(d+2) \leq -2$ $\quad$ (Lemma 2 §24.2)
$0 \leq d$

Remark: $H^0(\mathbb{P}^1, d[\infty]) = \begin{cases} 0 & \text{if } d<0 \\ \mathbb{C}[z]_{\leq d} & \text{if } d \geq 0 \end{cases}$

(if $d<0$: any $f \in H^0(\mathbb{P}^1, d[\infty]) \setminus \{0\}$ will have a zero at ∞ but no poles)

(if $d \geq 0$: $f \in \mathcal{M}(\mathbb{P}^1) \setminus \{0\}$ will at worst a pole at ∞ is rational & has no other pole $\Rightarrow f$ is a polynomial of degree $\leq d$ (# zeros = # poles $\leq d$))

② $E = \mathbb{C}/\Lambda$ & $p \in E$

Exercise: $|[p]| = \{[p]\}$ & $H^0(E, \mathcal{O}_p) = \{f : (f) \geq -[p]\} = \mathbb{C}$

Next, we write some properties for $|D|$ given any compact R.S. X

Proposition 2: (1) If $D \geq 0$, then $1 \in H^0(X, \mathcal{O}_D)$ & $D \in |D|$.

(2) If $\deg D < 0$, then $H^0(X, \mathcal{O}_D) = 0$ $\quad$ ($E \in |D|$ has $\underset{\geq 0}{\deg E} = \underset{< 0}{\deg D}$ Contr!)

(3) If $\deg D = 0$ $\quad H^0(X, \mathcal{O}_D) = \begin{cases} \mathbb{C} & \text{if } D \sim 0 \quad (D \text{ ppal}) \\ 0 & \text{if } D \not\sim 0 \end{cases}$

(4) If $\deg D = 1$ $\quad H^0(X, \mathcal{O}_D) = \begin{cases} 2 & \text{if } X \simeq \mathbb{P}^1 \\ \leq 1 & \text{else} \end{cases}$

Proof (3) $E \in |D|$, then $\deg E \geq 0$ & $E \sim D$ forces $\deg E = \deg D = 0$

$\Rightarrow E = 0$. This gives $|D| = \begin{cases} \{0\} & \text{if } D \sim 0 \\ \emptyset & \text{if } D \not\sim 0 \end{cases}$ because $E = 0$.

$\Rightarrow |D| = \mathbb{P}(H^0(X, \mathcal{O}_D)) = \begin{cases} pt & \text{if } D \sim 0 \\ \emptyset & \text{if } D \not\sim 0 \end{cases} \Rightarrow H^0(X, \mathcal{O}_D) = \begin{cases} \mathbb{C} & \text{if } D \sim 0 \\ \{0\} & \text{if } D \not\sim 0 \end{cases}$

[If $D \sim 0$, we get $H^0(X, \mathcal{O}_D) = H^0(X, \mathcal{O}) = \{ f \in \mathcal{M}(X) \mid (f) \geq 0 \} = \mathcal{O}(X) = \mathbb{C}$]

(4) Next Time

As part of our inductive proof of Riemann-Roch, we showed the following statement

Theorem: $H^0(X, \mathcal{O}_D) \leq \deg D$ if $X \not\cong \mathbb{P}^1$ & $H^0(\mathbb{P}^1, \mathcal{O}_D) = \deg D + 1$.
if $\deg D \geq 0$

Proof: Next Time