Lecture XXVII: Linear systems, maps to $\mathbb{P}^N$

Definition: • $|D| := \mathbb{P}(H^0(X, \mathcal{O}_D)) \simeq \mathbb{P}^{\dim H^0(X,\mathcal{O}_D)-1}$ (complete linear system assoc to D)

- Linear system = any projective subspace $\Lambda \subseteq |D|$

Lemma: $|D| \simeq \{E \in \mathrm{Div}(X) : E \geq 0 \ \& \ E \sim D\}$

$f \longmapsto \mathrm{div}_D(f) = (f) + D$

Example: $H^0(\mathbb{P}^1, d[\infty]) = \begin{cases} 0 & \text{if } d<0 \\ \mathbb{C}[z]_{\leq d} & \text{if } d \geq 0 \end{cases} \Rightarrow |d\infty| = \begin{cases} 0 & \text{if } d \leq 0 \\ \mathbb{P}^d & \text{if } d \geq 0 \end{cases}$

Proposition: (1) If $D \geq 0$, then $1 \in H^0(X, \mathcal{O}_D)$ & $D \in |D|$.

(2) If $\deg D < 0$, then $H^0(X, \mathcal{O}_D) = 0$ $\quad$ ($E \in |D|$ has $\underset{\overset{\vee}{0}}{\deg E} = \underset{\overset{\wedge}{0}}{\deg D}$ contn!)

(3) If $\deg D = 0$ $\quad H^0(X, \mathcal{O}_D) = \begin{cases} \mathbb{C} & \text{if } D \sim 0 \quad (D \text{ ppal}) \\ 0 & \text{if } D \not\sim 0 \end{cases}$

Next goals: ① Show that any compact R.S can be embedded as a sm. projective curve in some $\mathbb{P}^r$. We will do this via $|D|$ for $D \in \mathrm{Div}(X)$ with $\deg D \geq 2g+1$

② Classify compact R.S of low genera.

§ 27.1 More on linear systems:

Proposition: If $\deg D = 1$ $\quad H^0(X, \mathcal{O}_D) = \begin{cases} 2 & \text{if } X \simeq \mathbb{P}^1 \\ \leq 1 & \text{else} \end{cases}$ $\quad$ (example above $D = [\infty]$)

Proof: - $H^0(\mathbb{P}^1, \mathcal{O}_D) = H^0(\mathbb{P}^1, \mathcal{O}_{[\infty]}) = \mathbb{C}[z]_{\leq 1} \Rightarrow \dim = 2$.

• **Claim:** If $\dim H^0(X, \mathcal{O}_D) \geq 2 \Rightarrow X \simeq \mathbb{P}^1$.

Pf/ If $\dim \mathcal{O}_D(X) \geq 2 \Rightarrow \exists\, E \geq 0$ $E \sim D$ of $\deg = 1$ $\Rightarrow E = [p] \sim D$

$$\Rightarrow \dim H^0(X, \mathcal{O}_D) = \dim \overbrace{H^0(X, \mathcal{O}_{[p]})}^{=\text{ mero functions with a pole at } p \text{ of order} \leq 1 \ \& \text{ holo away from } p.} \geq 2$$

We always have $\mathbb{C} \subseteq H^0(X, \mathcal{O}_{[p]})$ so by dim reasons, $\exists\, f \in H^0(X, \mathcal{O}_{[p]}) \smallsetminus \mathbb{C}$

By construction, f has exactly one pole at $[p]$ of order 1, so $f: X \longrightarrow \mathbb{P}^1$ is holo non constant map of degree 1, hence a biholomorphism, ie $X \simeq \mathbb{P}^1$.

Corollary: $H^0(X,\mathcal{O}_{[p]}) = \mathbb{C}$ if $X \not\cong \mathbb{P}^1$.

Theorem: $H^0(X,\mathcal{O}_D) \leq \deg D$ if $X \not\cong \mathbb{P}^1$ & $H^0(\mathbb{P}^1,\mathcal{O}_D) = \deg D + 1$.
if $\deg D \geq 0$

For the proof, we'll need the following lemma:

Lemma: $\dim H^0(X,\mathcal{O}_{D-[p]}) \geq \dim H^0(X,\mathcal{O}_D) - 1$. $\forall\, p \in X$

Proof: Use the ses $0 \longrightarrow \mathcal{O}_{D-[p]} \longrightarrow \mathcal{O}_D \longrightarrow \mathbb{C}_p \longrightarrow 0$

to see $\mathcal{O}_D(X) / \mathcal{O}_{D-p}(X) \subseteq \mathbb{C}_p(X) = \mathbb{C}$ as a subspace, so

$$\dim\left(\mathcal{O}_D(X) / \mathcal{O}_{D-p}(X)\right) = \dim \mathcal{O}_D(X) - \dim \mathcal{O}_{D-p}(X) \leq 1$$

finite dimensional by RR

Proof of Theorem: Set $d = \deg D$

- If $X \cong \mathbb{P}^1$, then $D \sim d[\infty]$ & $H^0(X,\mathcal{O}_D) \cong H^0(X, d[\infty]) \cong \mathbb{C}[z]_{\leq d}$ which has basis $\}1, z, \dots, z^d\{$ if $d \geq 0$ & $= 0$ otherwise.
- Next, assume $X \not\cong \mathbb{P}^1$, pick $p \in X$ & consider the chain of inclusions of d vector sp.

$$H^0(X,\mathcal{O}_{D-(d-1)[p]}) \subseteq H^0(X,\mathcal{O}_{D-(d-2)[p]}) \subseteq \cdots \subseteq H^0(X,\mathcal{O}_{D-[p]}) \subseteq H^0(X,\mathcal{O}_D)$$

Since $\deg(D-(d-1)[p]) = 1$ & $X \not\cong \mathbb{P}^1$, Proposition above says

$$\dim H^0(X,\mathcal{O}_{D-(d-1)[p]}) \leq 1.$$

Since at each inclusion the dim grows at most by 1 (by the Lemma above), we get

$$\dim H^0(X,\mathcal{O}_D) \leq \dim H^0(X,\mathcal{O}_{D-(p-1)[p]}) + \underset{\substack{\uparrow \\ \#\text{ of inclusions}}}{(d-1)} \leq 1 + d - 1 = d.$$

§27.2 Maps To $\mathbb{P}^N$:

View $\mathbb{P}^N = \mathbb{C}^{N+1}\setminus\{0\} / \mathbb{C}^*$ as a complex manifold with the standard open cover

$$\mathbb{P}^N = \bigcup_{i=0}^{N} U_i \quad \text{where } U_i := \{\underline{x} = [x_0 : \dots : x_N] : x_i \neq 0\}$$

$$\begin{tikzcd} U_i \arrow[d, "\varphi_i"'] & \underline{x} \arrow[d, mapsto] \\ \mathbb{C}^N & \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_N}{x_i}\right) \end{tikzcd}$$

Clearly φ_i is homeomorphism $(\varphi_i^{-1}(\underline{y}) = [y_0 : \dots : y_{i-1} : 1 : y_{i+1} : \dots : y_N]$

Set X = compact RS

- Fix $f_0, \dots, f_N \in \mathcal{M}(X)$ which do not vanish identically on X. We define

$$\boxed{F = [f_0 : \dots : f_N] : X \longrightarrow \mathbb{P}^N}$$

as follows. Given $x \in X$, pick a local chart (V, z) with $V \underset{z}{\overset{\simeq}{\longrightarrow}} \mathbb{D}$, $x \mapsto 0$ & set

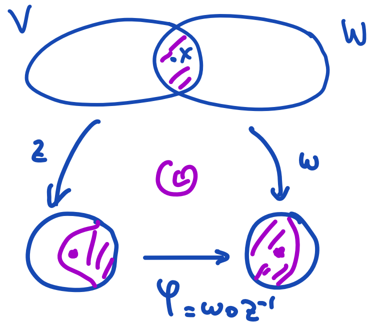
$k = \min_j \operatorname{ord}_x(f_j)$ Write $f_j|_V = z^k g_j(z)$ g_j holo near x

$\exists\, j$ with $g_j(x) \neq 0$ $\Rightarrow$ $F|_V(z) = [g_0(z) : \dots : g_N(z)] \in \mathbb{P}^N$

Lemma: This is chart independent.

Proof: Fix (V, z) & (W, w) coord charts at x

- k is chart indep ✓



$$\varphi \text{ biholo} \Rightarrow \begin{cases} \varphi(z) = z\,\psi(z) \\ \psi(0) \neq 0 \end{cases}$$

- $z^k g_j(z) = \underset{\substack{\downarrow \\ w = \varphi \circ z}}{w^k} h_j(w) = z^k \underbrace{\psi^k(z)\, h_j \circ \varphi(z)}_{= g_j(z)}$

$$\Rightarrow [g_0 : \dots : g_N] = [\underbrace{\psi^k(z)}_{\neq 0 \text{ near } x} h_0 \circ \varphi(z) : \dots : \psi^k(z)\, h_N \circ \varphi(z)]$$

$= [h_0(w) : \dots : h_N(w)]$ on an open in $V \cap W$

$(\psi^{-1}(\{0\}) \subseteq V \cap W)$ $\square$

By construction, F is holomorphic ($g_0, \dots, g_N$ are holomorphic!)

⚠ Issue in general: common poles & zeros of $f_0 \dots f_N$. These are called base points of $\{f_0, \dots, f_N\}$. We cannot get rid of them if $\dim X > 1$.
(next time!)

For $\dim X = 1$ we were able to bypass issues by removing these common poles & zeros (via factorizing out z^k)

Definition: F is non-degenerate if $\operatorname{Im} F$ is not contained in any projective proper subspace of $\mathbb{P}^N$.

Proposition: $\{f_0, \dots, f_N\}$ is linearly independent $\Leftrightarrow$ F is nondegenerate

Pf/ Proj subspaces lie in hyperplanes $\sum_{i=0}^{n} \lambda_i y_i = 0$ $\lambda_0, \dots, \lambda_N$ not all 0.

F degenerate $\Leftrightarrow$ $\exists\, \lambda_0, \dots, \lambda_N$ not all 0 with $\sum_{i=0}^{n} \lambda_i f_{i}(z) = 0$

$\Leftrightarrow$ $\{f_0, \dots, f_N\}$ linearly independent.

Example: Rational canonical curves

Consider the n+1 linearly independent meromorphic functions on $\mathbb{P}^1$:

$$f_0(z) = 1, \quad f_1(z) = z, \quad \dots, \quad f_N(z) = z^N$$

$$\Rightarrow \mathbb{P}^1 \xrightarrow{f = (f_0 : \dots : f_N)} \mathbb{P}^N \qquad (f(\infty) = [0 : \dots : 1])$$

$N = 1$: $f = id$

$N = 2$ $f(x:y) = [x^2 : xy : y^2]$ (on U_0 $z = \frac{y}{x}$.)

$\Rightarrow \operatorname{Im} f \subseteq (XZ - Y^2 = 0) \subseteq \mathbb{P}^2$.

Claim: $\operatorname{Im} f = (XZ - Y^2 = 0) \subseteq \mathbb{P}^2$

Pf/ We can check this by inverting F on $F^{-1}(U_i)$ for $i = 0, 1, 2$.

$i=0$: Pick $[X:Y:Z]\cap U_0$ so $\frac{Z}{X}=\left(\frac{Y}{X}\right)^2$ Set. $x=1$ & $y=Y$ to get $f(1:Y)=[X:Y:Z]$

$i=2$ is similar

$i=1$ set $Z=1$ so $Y^2=X$ If $X=0$ $\Rightarrow$ Pt is $[0:0:1]=F_{([0:1])}$

If $X\neq 0$, take $y=\sqrt{Y^2}$ by some branch of log $\Rightarrow F[\frac{Y}{y}:y]=[X:Y:1]$.
(sign won't matter: we get the same pt $[x:y]$ in $\mathbb{P}^1$) □

- Main examples of non-deg maps = those induced by $|D|$
- $\{f_0,\dots,f_N\}$ basis of $H^0(X,\mathcal{O}_D)$ ⇝ $F=\Phi_D : X \xrightarrow{|D|} \mathbb{P}^N$.

Lemma: Up to $\mathrm{Aut}(\mathbb{P}^N)$, Φ_D is independent of the basis

Proof:

$$\begin{tikzcd}
X \arrow[r, "(f_0,\dots,f_N)"] \arrow[dr, "(g_0:\dots:g_N)"'] & \mathbb{P}^N \arrow[d, "\sim"] \\
\arrow[ur, phantom, "\circlearrowleft"] & \mathbb{P}^N
\end{tikzcd}$$

Automorphism induced by change of basis matrix

Example above: $X=\mathbb{P}^1$, $D=N[\infty]$ $|D|\simeq \mathbb{C}[z]_{\leq N}$.

Q1: When is F local homeo onto its image ("immersion")?

A: When F separates tangents ($\forall\, x\in X$ $\exists\, j$ with $x\in F^{-1}(U_j)$ & $dg_j(x)\neq 0$)

Q2: When is F an embedding (injective + immersion)?

A: F separates points ($\equiv$ injective) & separates tangents.

Q3: Can we determine these conditions from D? A: We'll give sufficient conditions.

§27.3 What Ds make Φ_D an embedding?

Theorem 1: Fix X compact R.S. of genus g & $D \in \mathrm{Div}(X)$. Fix $\{f_0, \ldots, f_N\}$ basis for $H^0(X, \mathcal{O}_D)$ & $\Phi_D : X \xrightarrow{|D|} \mathbb{P}^N$. If $\deg D \geq 2g+1$, then Φ_D is an embedding.

- For the proof, we need the notion of globally generated sheaf $\mathcal{O}_D$.

Definition: We say $\mathcal{O}_D$ is globally generated if for every $x \in X$ $\exists f \in \mathcal{O}_D(X)$ with $\mathcal{O}_{D,x} = \mathcal{O}_{X,x} \cdot (f)$. ie, any germ $\varphi \in \mathcal{O}_{D,x}$ has the form $\varphi = \psi \cdot f$ with ψ holo around x.

Lemma 1: $\mathcal{O}_{D,x} = \mathcal{O}_{X,x}(f) \iff \mathrm{ord}_x(f) = -D(x)$

Proof: $\mathcal{O}_{D,x} = \{ g = \sum_{n=k}^{\infty} a_n z^n \in \mathbb{C}((z)) \}$ with $k = -D(x)$.

$f \in \mathcal{O}_{D,x}$ generates $\iff a_{-k} \neq 0 \iff \mathrm{ord}_x f = -D(x)$.

Theorem 2: Fix X compact R.S. of genus g & $D \in \mathrm{Div}(X)$ with $\deg D \geq 2g$. Then, $\mathcal{O}_D$ is globally generated.

Proof: Fix $x \in X$ & build $f \in \mathcal{O}_D(X)$ with $\mathrm{ord}_x(f) = -D(x)$.

Set $D' = D - [x]$

By Theorem §24.1: $\deg D' = \deg D - 1 \geq 2g - 1 > 2g-2 \Rightarrow H^1(X, \mathcal{O}_{D'}) = 0$

$\deg D \geq 2g > 2g-2 \Rightarrow H^1(X, \mathcal{O}_D) = 0$

RR says
$$\dim H^0(X, \mathcal{O}_D) = 1 - g + \deg D = 1 - g + \deg D' + 1$$
$$= \dim H^0(X, \mathcal{O}_{D'}) + 1 > \dim H^0(X, \mathcal{O}_{D'})$$

$\Rightarrow \exists f \in H^0(X, \mathcal{O}_D) \setminus H^0(X, \mathcal{O}_{D'})$, so $\mathrm{ord}_x(f) = -D(x)$ $\square$

Corollary: $\mathcal{O}_D$ is globally generated $\iff$ $\dim H^0(X,\mathcal{O}_{D-[p]}) = \dim H^0(X,\mathcal{O}_D)$ $\forall p \in X$.

(This is equiv. to $\exists\, f \in H^0(X,\mathcal{O}_D) \setminus H^0(X,\mathcal{O}_{D-[p]})$)

Proof of Theorem 1: Need to show Φ_D separates points & tangent vectors

- Φ_D is injective: Pick $x_1 \neq x_2$ in X & show $\Phi_D(x_1) \neq \Phi_D(x_2)$

Define $D' = D - [x_2]$ Note: $\deg D \geq 2g+1 > 2g$ & $\deg D' \geq 2g$

By Theorem 2, $\mathcal{O}_{D'}$ & $\mathcal{O}_D$ is globally generated.

Pick $f \in H^0(X,\mathcal{O}_{D'})$ with $\operatorname{ord}_{x_1}(f) = -D'(x_1) = -D(x_1)$

- $f \in H^0(X,\mathcal{O}_{D'})$ so $\operatorname{ord}_{x_2}(f) \geq -D'(x_2) = -D(x_2)+1 > -D(x_2)$

$\operatorname{ord}_x(f) \geq -D'(x) = -D(x)$ $\forall x \neq x_2$

$\Rightarrow f \in H^0(X,\mathcal{O}_D) = \langle f_0, \dots, f_N \rangle$

Pick $\lambda_0, \dots, \lambda_N \in \mathbb{C}$ with $f = \sum_{j=0}^{N} \lambda_i f_i$

- Pick 2 charts (V_1, z_1), (V_2, z_2) around x_1 & x_2 with

$$\begin{tikzcd} V_1 \arrow[r, "\cong", "z_1"'] & \mathbb{D} \\ x_1 \arrow[r, mapsto] & 0 \end{tikzcd} \quad , \quad \begin{tikzcd} V_2 \arrow[r, "\cong", "z_2"'] & \mathbb{D} \\ x_2 \arrow[r, mapsto] & 0 \end{tikzcd}$$

Claim: $k_i = \min_j \operatorname{ord}_{x_i}(f_j) = -D(x_i)$ for $i = 1,2$

Pf/ $\operatorname{ord}_{x_i}(f_j) \geq -D(x_i)$ $\forall j$ because $f_j \in H^0(X,\mathcal{O}_D)$.

$\mathcal{O}_D$ globally generated $\Rightarrow \exists\, h_i \in \langle f_1, \dots, f_N \rangle$ with $\operatorname{ord}_{x_i}(h_i) = -D(x_i)$

$\operatorname{ord}_{x_i} h_i \geq k_i$ by construction, so $k_i = -D(x_i)$ $\square$

Write $f_j = z_1^{k_1} g_j^{(1)} = z_2^{k_2} g_j^{(2)}$ $\qquad g_j^{(i)} \in \mathcal{O}(V_i)$

$f = z_1^{k_1} h_1 = z_1^{k_2} h_2$ $\qquad h_1 \in \mathcal{M}(V_1)$, $h_2 \in \mathcal{M}(V_2)$

$$\begin{array}{ll} \bullet\ \mathrm{ord}_{x_2} f \geqslant -D(x_2)+1 = k_2+1 \Rightarrow \\ \bullet\ \mathrm{ord}_{x_1} f = -D(x_1) = k_1 \Rightarrow \end{array} \boxed{\begin{array}{l} h_2 \in \mathcal{O}(V_2)\ \&\ h_2(x_2)=0 \\ h_1 \in \mathcal{O}(V_1)\ \&\ h_1(x_1) \neq 0 \end{array}} \quad (*)$$

By construction $\bullet$ $\phi_D|_{V_i} = [g_0^{(i)} : \cdots : g_N^{(i)}]$ $\quad i=1,2$

$$\bullet \sum_{j=0}^{N} \lambda_j \underbrace{g_j^{(i)}(x_i)}_{f_j/z_i^{k_i}} = \underbrace{h_i(x_i)}_{= f/z_i^{k_i}}$$

If $\phi_D(x_1) = \phi_D(x_2)$, we get $[g_0^{(1)}(x_1) : \cdots : g_N^{(1)}(x_1)] = [g_0^{(2)}(x_2) : \cdots : g_N^{(2)}(x_2)]$ in $\mathbb{P}^N$

ie $\exists\ \alpha \in \mathbb{C}^*$ with $g_j^{(1)}(x_1) = \alpha\, g_j^{(2)}(x_2)$ for $j=0,\dots,N$

$$\Rightarrow \sum_{j=0}^{N} \lambda_j g_j^{(1)}(x_1) = h_1(x_1)$$

$$\underset{\overset{\|}{0}}{\alpha h_1(x_1)} \text{ by } (*) = \sum_{j=0}^{N} \lambda_j g_j^{(2)}(x_2) = \underset{\overset{\#}{0}}{h_2(x_2)} \text{ by } (*) \qquad \text{Contra!}$$

- ϕ_D <u>is an immersion:</u> Next Time (we'll use $D' = D - [x_0]$ for $x_0 \in X$)